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UNDERSTANDING FRACTIONAL CALCULUS

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Understanding Fractional Calculus

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PREFACE

The field of fractional calculus, though rooted in centuries-old mathematical inquiry, has only recently gained significant momentum across diverse scientific and engineering disciplines. The concept of extending differentiation and integration to non-integer (fractional) orders was once a theoretical curiosity, but it has now emerged as a powerful tool for modeling memory, hereditary systems, anomalous diffusion, and complex dynamical behaviors in real-world phenomena.

This book, titled *Understanding Fractional Calculus*, aims to offer a clear, structured, and accessible introduction to this fascinating area of mathematical analysis. It is intended for students, researchers, and professionals who are curious about fractional calculus or are seeking to apply it within their respective fields.

The book is organized into five carefully curated chapters:

Chapter I: Introduction to Fractional Calculus

This chapter provides the foundational motivation and historical background of fractional calculus. It introduces the basic definitions and outlines the evolution of the subject from pure mathematics to modern-day applications.

Chapter II: Special Functions

Fractional calculus is deeply interwoven with a wide class of special functions, such as the Gamma function, Mittag-Leffler function, and hypergeometric functions. This chapter explains these functions and their roles in defining and solving fractional differential equations.

Chapter III: Different Approaches of Fractional Calculus

Multiple definitions and perspectives exist within the fractional calculus community. This chapter explores various formalisms—including Riemann-Liouville, Caputo, and Grünwald-Letnikov derivatives—highlighting their similarities, differences, and areas of application.

Chapter IV: Extended Transforms and Fractional Differential Equations

This chapter delves into how classical transforms, like the Laplace and Fourier transforms, are extended to handle fractional orders. It also discusses methods for solving fractional differential equations using these extended tools.

Chapter V: Applications of Fractional Calculus

To illustrate the practicality and versatility of fractional calculus, this chapter presents selected applications in physics, control theory, bioengineering, finance, and other fields. Real-world case studies demonstrate how fractional models provide a more accurate description of complex systems.

Our goal in presenting this book is to demystify fractional calculus and encourage further exploration and research. Each chapter builds upon the previous one, providing a coherent and comprehensive journey through theory, tools, and applications.

We hope that this book serves as a valuable resource for your learning, teaching, or research endeavors in fractional calculus and its many applications.

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CHAPTER - I

INTRODUCTION TO FRACTIONAL CALCULUS

1.1 Introduction:

In general, generalization is overruled on restrictions. One such generalization in mathematics is fractional calculus. Thus fractional calculus is the generalization of ordinary/ classical calculus. Objectives of this chapter are to study origin of fractional calculus, to study historical development of fractional calculus right from beginning of field till to date. Also to make some conclusion on the studied part of chapter and at last, chapter wise summary of the thesis prescribed.

1.2 Fractional Calculus:

In mathematics generalization or extension of knowledge creates lots of research in the subject which is very useful for later development of mathematics as well as that particular topic/subject. One such very important generalization of classical calculus is the fractional calculus. It may worth to say fractional calculus is the interpolation of classical calculus.

So, fractional calculus is the extension or generalization of traditional calculus that is fractional calculus is the derivative and integration of arbitrary order rational, irrational or complex. Actually work on fractional calculus begins very close with the beginning of work on traditional calculus. Physical and geometrical interpretation of traditional calculus gives rise in its fast development where as inability to represent physical and geometrical interpretation of fractional calculus results in very slow development. In 17^{th} century traditional calculus begins and spreads very fast with its application where as fractional calculus though it begins very close traditional calculus in 17^{th} century negligible development found up to the 20^{th} century. In last two to three decades we found some notable development of fractional calculus.

While developing the study of classical calculus i.e. differential calculus and integral calculus question may arise that 'what will be fractional order derivative and fractional order integrals?' but the written proof to this question was found in communication letters between great mathematicians Guillaume L'Hôpital and Gottfried Leibniz on 30^{th} September 1695. Answer given by great mathematician Gottfried Leibniz on 30^{th} September 1695 to the question raised by another great mathematician Guillaume L'Hôpital through letter to the question "What will be the meaning of $\frac{d^n f}{dx^n}$, if n=1/2?" as **"This is an apparent paradox from which one day a useful consequence will be drawn"**. After 30th September 1695 mathematicians who are working on classical calculus with special case of differentiation and integration of arbitrary order named the field as **Fractional calculus.**

Thereafter, Lacorix, L'Hôpital, Holmgren, Euler, Letnikov, Grünwald, Lagrange, Krug, Laplace, Riemann, Liouville, Heaviside, Lourant, Hadamard, Hardy, Littlewood, Weyl, Erdelyi, Kober, Widder, Osier, Sneddon, Mikolas, Al-Bassam worked and developed the basic concepts of fractional calculus.

The basic and theoretical investigation was first carried out by Laplace in 1812 defined fractional derivative in terms of integral, in 1819 Lacorix determine the derivative of arbitrary order and generalize it, in 1822 Fourier defined fractional order derivative in terms of integral representation. After Fourier, Liouville (1832a) where defined the first outcast of an operator of fractional integration. Then Swedish mathematician Holmgren (1864), Euler (1865) took the first step in the study of fractional integration. Today there exist many different forms of fractional derivative and fractional integral operators. The Riemann-Liouville operator is the most frequently used usually for fractional integration. Caputo, in the year 1967, defined a useful formula to obtain fractional derivative of a function.

More than 300 years the appearance of the fractional integration and differentiation concepts/ideas act as a theoretical since there were unavailability of geometrical and physical interpretation of these concepts/ideas.

F.Ben Adda, has suggested a different approach to geometrical interpretation of fractional integration and fractional differentiation, based on the idea of the contact of *αth* (non-integer) order. However, without visualization it is difficult to speak about an acceptable geometrical interpretation. I. Podlubny in the year 2002 represented the geometrical interpretation of the integral and the fractional integral. Like other mathematical concepts, the development of fractional calculus has passed through various disagreements, inaccuracies, farces, etc. That sometimes made mathematicians distrusting in the general concept of fractional operators.

Development in application of fractional calculus was first discovered by Neils Henrik Abel in 1923, he applied fractional calculus to solve the problem in physics for finding tautochrone curve (tautochrone curve is the problem of finding shape of curve for frictionless point mass particle in time decent under the uniform gravitational force is independent of starting point).

Applications of fractional calculus are found in various diversified fields of science and engineering, such as diffusion, anomalous diffusion, fluid flow, relaxation, oscillation, rheology, reaction-diffusion, turbulence, diffusive transport akin to diffusion, electric networks, polymer physics, chemical physics, electro chemistry of corrosion, Economics, relaxation processes in complex systems, biological sciences, signal processing, biophysics, polymers, statistical mechanics, statistics, probability, transport theory, control theory, Electrical circuit, elasticity, potential energy and others.

1.3 Historical Development of Fractional Calculus:

Though the work on fractional calculus begins very close to the beginning of work on traditional calculus but the actual proof found in communication letter between the great mathematicians Gottfried Leibniz and Guillaume L'Hôpital on 30th September 1695. This date 30th September 1695 therefore treated as birth date of Fractional Calculus. The concept of communication was the question "what will be the meaning of a derivative of integer order $d^n y/dx^n$ when n becomes fraction?" as it is found affirmative after long duration of existence of problem then again question arrives on n that "Whether n becomes rational, irrational or complex?" as it answered affirmatively again after long duration, hence the name Fractional Calculus misnomer. Fractional calculus can also be referred as differentiation and integration of arbitrary order, also be referred as generalization of traditional calculus. However in 1738 Euler's question in this regard as d^n can be found by successive (continued) differentiation if n is positive integer then how and what will be found if n becomes fractional?

In 1819 first written work appears concerned with fractional calculus in a text by S. F. Lacroix where he develops a formula for factional derivative of function $y = x^m$, m a positive integer.

$$D\,y = m\,x^{m-1}$$

$$D^2\,y = m(m-1)\,x^{m-2}$$

$$\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots$$

$$D^n\,y = m(m-1)\ldots(m-n)\,x^{m-n},\quad n \le m.$$

This can be written in factorial notation as

$$D^n\, y = \frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}\, x^{m-n} \tag{1}$$

Replacing factorial notation by Legendre's symbol (Gamma function) Lacroix obtained

$$D^n\, y = \frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)}\, x^{m-n} \tag{2}$$

Using this formula he obtained one half derivative of the function y= x, in equation (1.3.2) put m= 1 and n= ½ we have

$$D^{\frac{1}{2}}\, y = \frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \frac{\Gamma(1+1)}{\Gamma\left(1-\frac{1}{2}+1\right)}\, x^{1-\frac{1}{2}}$$

$$D^{\frac{1}{2}}\, x = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}\, x^{\frac{1}{2}}$$

$$D^{\frac{1}{2}}\, x = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{3}$$

Thus one half derivative of x was obtained.

In 1848 William Centre evaluated fractional derivative of constant using Lacroix definition where he takes zero power of x so that x^0=1 a constant then

$$D^{\frac{1}{2}}\, 1 = D^{\frac{1}{2}}x^0 = \frac{1}{\Gamma(\frac{1}{2})}x^{-\frac{1}{2}} \neq 0 \tag{4}$$

Which shows one half derivative of constant is other than zero which does not satisfy traditional property of derivative because of such lacunas (as per traditional derivative) study on fractional calculus progressed slowly. The method of Lacroix using linearity of fractional derivatives is applicable to any analytic function by term wise differentiation of its power series expansion. In order for the method to be considered general, this class of functions is too narrow has very limited scope.

Next to S. F. Lacroix, Joseph B. J. Fourier in 1822 evaluated derivatives of fractional order or arbitrary order. He obtained definition of fractional calculus from integral representation of function f(x) as

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, dt \int_{-\infty}^{\infty} \cos p(x-t)\, dp \tag{5}$$

Now for an integer n

$$\frac{d^n}{dx^n}\cos p(x-t) = p^n \cos\left[p(x-t) + \frac{1}{2}n\pi\right]$$

By replacing n with α arbitrary any quantity positive or negative, He obtained definition for fractional evaluation

$$\frac{d^{\alpha}}{dx^{\alpha}} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha)\, d\alpha \int_{-\infty}^{\infty} p^{n} \cos\left[p(x-\alpha) + \frac{1}{2}\alpha\pi\right] dp \qquad (6)$$

Contribution to theoretical development of fractional calculus also given by Euler, Leibniz, Laplace, P. Kelland, Greer etc

Simultaneously with these initial theoretical developments, first practical applications of fractional calculus found or discovered by Niels Henrik Abel in 1823. Abel considered the solution of the integral equation related to the tautochrone problem (tautochrone is the problem of finding shape of curve for frictionless point mass particle in time decent under the uniform gravitational force is independent of starting point). If the time of slide is a known constant k he found the solution via an integral equation given by

$$k = \int_0^x (x-t)^{-\frac{1}{2}} f(t)\, dt \qquad (7)$$

This integral is a particular case of definite integral which defines fractional integral of order 1/2, and written as

$$k = \sqrt{\pi}\left[\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\right] f(x) \qquad (8)$$

and applied one half derivative

$$\left[\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right] k = \sqrt{\pi}\, f(x) \qquad (9)$$

Thus after evaluating one half derivative of k required function f(x) obtained. This is an great success in the field of fractional calculus by Abel. Again we come to mathematical controversy that fractional derivative of a constant is not always zero.

Results obtained by S. F. Lacroix, Joseph B. J. Fourier, Niels Henrik Abel etc attracted many mathematicians towards fractional calculus.

In 1832 Joseph Liouville, who made the first major study of fractional calculus,[8]. He began to study fractional calculus to apply his results to problems in potential theory. Joseph Liouville began his theoretical development using the result for derivatives of integer order *n*

$$D^{n} e^{ax} = a^{n} e^{ax} \qquad (10)$$

This equation (1.3.5) easily generalized for arbitrary order α as

$$D^{\alpha} e^{ax} = a^{\alpha} e^{ax} \qquad (11)$$

He applied this result of arbitrary derivative of exponential function f(x) to the series given by

$$f(x) = \sum_{n=0}^{\infty} c_n \, e^{a_n x}, \quad \text{Re } a_n > 0 \qquad (12)$$

As

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n \, a_n^{\alpha} \, e^{a_n x} \qquad (13)$$

Where arbitrary order α is any number rational, irrational or complex this is Liouville's first formula for fractional calculus. As it is used only for particular type of function so it has such disadvantage, again he work on the topic and obtained second formula, for which he begins with definite integral correspond to gamma function

$$I = \int_0^{\infty} t^{a-1} \, e^{-xt} \, dt \qquad a > 0, x > 0 \qquad (14)$$

On substituting xt = u, obtain

$$I = x^{-a} \int_0^{\infty} u^{a-1} \, e^{-u} \, du$$

$$I = x^{-a} \, \Gamma(a)$$

$$x^{-a} = \frac{1}{\Gamma(a)} \, I$$

Then by applying operator D^{α}

$$D^{\alpha} x^{-a} = D^{\alpha} \left[\frac{1}{\Gamma(a)} \, I \right]$$

$$D^{\alpha} x^{-a} = \frac{(-1)^{\alpha}}{\Gamma(a)} \int_0^{\infty} t^{a+\alpha-1} \, e^{-xt} \, dt$$

$$D^{\alpha} x^{-a} = \frac{(-1)^{\alpha} \Gamma(+\alpha)}{\Gamma(a)} \, x^{-a-\alpha}, \qquad a > 0 \qquad (15)$$

Equation (15) is the second formula discovered by Liouville which were too narrow as like first, both formulas are used only for particular type of functions. First is used for the function in (12) and second one used only for function of type x^{-a} with a > 0. All the previous definitions are not suitable for a general definition of fractional derivative. Many mathematicians worked on and emerged similar type of formulas. As like above formulas H. R. Greer developed to find fractional derivative of trigonometric functions. In this study of fractional calculus in some extent mathematicians obtained result but including them self no one satisfied fully, which is the reason progress in the development of fractional calculus found very slow. Though the two definitions derived by Liouville in account of fractional calculus which has limited scope in the field then also he does not leave the field but go on working and again one more diamond included in his favor that he is the first to attempt to solve differential equation having fractional order differentials. In 1833 George Peacock, in 1839, S.S. Greatheed and many more mathematicians also worked on fractional differential equations. George Peacock supported Lacroix's version and erroneous in many points remark placed on Liouville's version. In 1839 and1846 P. Kelland published two works and supported Liouville's version. In 1840 Augustus De Morgan denies to support both the versions of Lacroix and Liouville.

In student days G. F. Bernhard Riemann works on fractional calculus. He developed theory of fractional integration which was published later in 1892. Riemann generalizes Taylor series and obtained the formula in fractional integral form as

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}\, f(t)dt + \phi(t) \tag{16}$$

Where c and x are limits of integration and ϕ(x) is a complementary function.

In 1880 A. Cayley makes a comment on Riemann's equation in (16) related to difficulty in equation concerned with complementary function containing infinity of arbitrary constants.

At the end of nineteenth century due to erroneous versions of definition of fractional calculus there was long duration dispute between mathematicians which definition Lacroix-Peacock or Liouville to be used or correct. Riemann also entangled in his version due to complementary function. Due to such situation mathematicians in that period on theory of fractional operator had a general distrust.

Generalized Leibniz's nth derivative of a product found in many modern applications B. Ross 1975. Liouville and after some years C. J. Hargreave worked on generalization of Leibniz product rule.

$$D^{\alpha} f(x)g(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D^{n}f(x)D^{\alpha-n}g(x) \tag{17}$$

Where $\binom{\alpha}{n}$ the generalized binomial coefficient given by

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha-n+1)}$$

In 1858 H. R. Greer [11] worked on finite differences of fractional operator which is again improved by W. Zachartchenxo in 1861. In 1868 monograph on application of fractional calculus to the solution of ordinary differential equation given by H. Holmgren where in introduction part he declare that the work of Liouville and Spitzer were too restrictive and his aim is to find complete solution without any restrictions.

Support on development to fractional calculus also given by N. Ya. Sonin, his work begins with cauchy's integral formula and work is entitled by "On differentiation with arbitrary index". A. V. Letnikov gives extension to N. Ya. Sonin's works, wrote four papers on the topic . Both are tried to generalize the nth derivative of cauchy's integral formula in which by factorial of an arbitrary number can be generalized easily by factorial gamma function but integrand of formula unable to evaluate if n is arbitrary and hence both of them unable to get complete success in achievement of target. In 1884 H. Laurent [9] worked on same topic i.e. to generalize cauchy's integral formula where his considered contour was an open circuit on a Riemann surface where as N. Ya. Sonin and A.V. Letnikov considered counter was a closed circuit. These methods of contour integration developed the definition for arbitrary order.

$$ {}_cD_x^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_c^x (x-t)^{\alpha-1}\ f(t)dt \qquad \text{Re } \alpha > 0 \qquad (18) $$

Riemann definition (1.3.16) without complementary function coincides with this definition when x > c. Here if c=0 then the definition is referred as Riemann-Liouville fractional integral a widely used version of fractional calculus given by

$$ {}_0D_x^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}\ f(t)dt \qquad \text{Re } \alpha > 0 \qquad (19) $$

In 1888 P. A. Nekrassov In 1890 A. Krug and in 1917 Weyl also derived the formula from cauchy's integral formula.

In the last decade of the nineteenth century Oliver Heaviside showed how certain linear differential equation solved using generalized operators by publishing number of papers. He is unable to justify his procedure though the results were correct. After long time, his procedure was justified by T. J. Bromwich in 1919. Work on theoretical development as well as application on fractional calculus was widely spread in the last three decades.

Parallel to theoretical development of fractional calculus development in its application also grows in different field of education. In last three to four decades we find numerous applications in all most all fields. Due to non-locality of fractional calculus, in many respects fractional calculus differs from the ordinary calculus. The chain rule and product rule take difficult form in case of fractional calculus. Like ordinary calculus, the fractional calculus does not have physical and geometrical meaning, though some hard slog has been made in this track recently. Additionally there are several different definitions of fractional calculus are found throughout the literature. Hence many mathematicians were distrustful and many mathematicians though they confuse then also they continued their work in the field.

During the last two decade Fractional Calculus has been applied to almost every field of Mathematics, Science and Engineering. It is now become conscious that the non-locality is not a negative aspect but it leads fractional calculus to model many natural occurrence containing long memory. Examples of such uncharacteristic systems are plentiful in nature. We list few of them: network traffic, cellular diffusion processes, dynamics of viscoelastic materials, atmospheric diffusion of pollution etc. All such systems have non-local dynamics involving long memory which cannot be reproduced using ordinary calculus. In fact ordinary calculus models the ideal behavior and Fractional calculus models the real behavior. Therefore fractional differential equations are helpful for the modeling of many asymmetrical phenomena in nature and in multifaceted systems.

Development of Fractional calculus has found intensively from 1974 when the first international conference especially in the field of fractional calculus took place at the University from New Haven, Connecticut in 1974. Which was organized by Bertram Ross approximately 94 mathematicians contributed in the conference and its proceeding contains 26 papers on the topic at the time. Second international conference on fractional calculus was conducted by Adam Mc Bride and Garry Roach at University of Strathclyde, Glasgow, Scotland in 1984. Third international conference especially on fractional calculus was took place at Nihon University, Tokyo, Japan in 1989 by Katsuyuki Nishimoto. Fourth international conference organized by Peter Rusev, Ivan Dimovski and Virginia Kiryakova at Varna, Bulgaria in 1996. After 1996 number international workshops, symposium was held on fractional calculus, special functions and its applications. In 23-25 June 2014 next international conference held especially on fractional calculus and its applications at Catania in Italy. The special session "Fractional Calculus: Quo Vadimus? (Where are we going?)" was proposed and reviewed by F. Mainardi.

The idea to have open sessions and discussions on Open problems in fractional calculus was realized from the first conference chaired by the late Professor B. Ross (one of the pioneers of fractional calculus in the contemporary era). Such open problems sessions took places also in the next international conferences on fractional calculus.

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CHAPTER -II

SPECIAL FUNCTIONS

2.1 Introduction

Special functions plays very crucial role in the development of fractional calculus. There are lots of special functions available in literature among them many special functions are related with fractional calculus, for solving problems consisting of fractional operators and fractional differential equations. Again from these related special functions only those which are closely associated with fractional calculus are studied.

Objectives of this chapter are to study some special functions & there properties in concern with fractional calculus. After Introduction section 2.1, section 2.2 we take an overlook about special functions and its progress. Section 2.3 deals in detail with gamma function, its properties and application also we study extended gamma functions. In next section 2.4 overview of beta function its properties, applications and extended form given thoroughly. Section 2.5 contains Mittag Leffler function of one parameter, two parameter and multi parameter with its properties and applications. Conclusion of chapter 2 was given in section 2.7.

2.2 **Special Functions:**

Special functions are a real or complex valued functions of one or more real or complex variables which are defined in such a way that its numerical values be obtained, evaluated or tabulated. We know simple functions like logarithmic function log(x); algebraic functions x^n, trigonometric functions sin(x), tan(x), exponential function e^x are belongs to the category of elementary functions. Higher functions both transcendental and algebraic come under the grouping of special functions. The study of special functions grew up with the calculus and it is therefore one of the oldest branches of analysis. It majorly developed in the nineteenth century as part of the theory of complex variables.

In the second half of the twentieth century it has acknowledged a new impetus from a relation with Lie groups and a relation with averages of elementary functions. The history of special functions is closely tied to the problem of terrestrial and celestial mechanics that were work out in the eighteenth and nineteenth centuries, the boundary-value problems of electromagnetism and heat in the nineteenth, and the eigen value problems of quantum mechanics in the twentieth.

Seventeenth-century England was the hometown of special functions. John Wallis at Oxford took first stepladder towards the theory of the gamma function long before Euler arrived at it.

Wallis had also the first come crossways with elliptic integrals while using Cavalieri's primitive predecessor of the calculus. A more refined calculus, which made promising the real flowering of special functions, was urbanized by Newton at Cambridge and by Leibnitz in Germany at some point in the period 1665-1685. Taylor's theorem was establish by Scottish mathematician Gregory in 1670, even though it was not available until 1715 after rediscovery by Taylor. In 1703 James Bernoulli solved a differential equation by an infinite series which would now be called the series representation of a Bessel function.

Even though Bessel functions were met by Euler and others in a variety of mechanics problems, no methodical study of the functions was completed until 1824, and the most significant achievements in the eighteenth century were the gamma function and the assumption of elliptic integrals. Euler establishes most of the major properties of the gamma functions around 1730. In 1772 Euler expected the Beta-function integral in stipulations of the gamma function. Only the duplication and multiplication theorems stay at the back to be discovered by Legendre and Gauss, respectively, early in the next century. Other significant developments were the discovery of Vandermonde's theorem in 1772 and the definition of Legendre polynomials and the discovery of their addition theorem by Laplace and during 1782-1785 by Legendre.

The fair-haired age of special functions, which was centered in nineteenth century German and France, was the result of developments in both mathematics and physics: the theory of analytic functions of a complex variable on one hand, and on the other hand, the field theories of physics (e.g. heat and electromagnetism) which requisite solutions of partial differential equations containing the Laplacian operator. The discovery of elliptic functions (the inverse of elliptic integrals) and their property of double periodicity were published by Abel in 1827. Elliptic functions grew up in symbiosis with the general theory of analytic functions and flourished throughout the nineteenth century by Jacobi and Weierstrass especially.

A further major development was the theory of hyper geometric series which set in motion in a systematic way with Gauss's memoir on the 2F1 series in 1812, a memoir which was a landmark also on the path towards rigor in mathematics. The 3F2 series was considered by Clausen (1828) and the iFi series by Kummer (1836). The functions which Bessel considered in his memoir of

1824 are oFi series; Bessel started from a problem in orbital mechanics, but the functions have found a place in every branch of mathematical physics.

The subject was considered to be branch of pure mathematics up to 1900, applied mathematics around 1950. In physical science special functions gained supplementary importance as solutions of the Schrodinger equation of quantum mechanics, but there were important developments of a purely mathematical nature also. In 1907 Barnes used gamma function to develop a new theory of Gauss's hyper geometric functions 2F1. Various generalizations of 2F1 were introduced by Horn, Kampe de Feriet, MacRobert, and Meijer.

2.3 Gamma Function:

Swiss mathematician Leonhard Euler (1707 – 1783) invented the gamma function. Gamma function is most frequently related with fractional calculus. The gamma function belongs to the special transcendental function category. In the integer-order calculus the factorial plays an important role because it is one of the most fundamental combinatorial tools. At the heart of the theory of special functions lies the Gamma function, in that nearly almost all of the classical special functions can be evaluated by this powerful function. Gamma functions have explicit series and integral functional representations, and thus provide ideal tools for establishing useful products and transformation formulae. In addition, applied problems frequently require solutions of a function in terms of parameters, rather than merely in terms of a variable, and such a solution is perfectly provided for by the parametric nature of the Gamma function. As a result, the Gamma function can be used to evaluate physical problems in diverse areas of applied mathematics. While the Gamma function's original intent was to model and interpolate the factorial function, mathematicians have discovered and developed many other interesting applications thus playing a particularly useful role in applied mathematics. Equations involving Gamma functions are of great interest to mathematicians and scientists, and newly proven identities for these functions helpful in finding solutions for many differential and integral equations. There exist a vast number of such identities, representations and transformations for the Gamma function, the comprehensive text providing over 400 integral and series representations for these functions. Gamma functions thus provide a rich field for ongoing research, which continues to produce new results. In 1959, in, It was stated that "of the so-called 'higher mathematical functions', the Gamma function is undoubtedly the most fundamental". For instance the rising factorial provides a direct link between the Gamma and hyper-geometric functions, and most hyper-geometric identities can be more elegantly expressed in terms of the Gamma function. In, it is stated clearly that, "the Gamma function and beta integrals are essential to understanding hyper-geometric functions." It is thus enlightening and rewarding to explore the various representations and relations of the Gamma function. The Gamma function has the same importance in the fractional-order calculus and it is basically given in the form of integral. The definitions, properties and well known examples of the Special Functions can be found in [8]. Euler's integral of second kind i.e. Euler's Gamma function, for z ϵ C \ {0, -1, -2, -3 …} in integral form be defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t}\, t^{z-1}\; dt, \qquad Re(z) > 0 \tag{2.3.1}$$

Limit form of Gamma function is obtained by substituting

$$e^{-t} = \lim_{n\to\infty}\left(1-\frac{t}{n}\right)^{n}$$

And then applying n times integration by part we get

$$\Gamma(z) = \lim_{n\to\infty}\left\{\frac{n!\; n^{z}}{z\,(z+1)\,(z+2)\,(z+3)\ldots.\,(z+n)}\right\} \tag{2.3.2}$$

And in product form it is defined as

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\} \tag{2.3.3}$$

Properties of Gamma function:

Theorem (2.3.1): (Recurrence Formula) For any z ϵ C \ {0, -1, -2, -3 …} prove that

$$(z+1) = z\,\Gamma(z)$$

Proof: We know

$$\Gamma(z) = \int_0^{\infty} e^{-t}\; t^{z-1}\; dt$$

Integration by parts

$$\Gamma(z) = \left[\frac{t^{z} e^{-t}}{z}\right] + \int_0^{\infty}\frac{t^{z} e^{-t}}{z}\; dt$$

$$\Gamma(z) = \int_0^{\infty}\frac{t^{z} e^{-t}}{z}\; dt$$

$$\Gamma(z) = \frac{1}{z}\int_0^{\infty} t^{z} e^{-t}\; dt$$

Hence, $(z+1) = z\,\Gamma(z)$.

Theorem (2.3.2): For n = 1, 2, 3, …… prove that $\Gamma(n + 1) = n!$.

Proof: We know

$$\Gamma(n) = \int_0^\infty e^{-t}\, t^{n-1}\, dt$$

Put n = 1, we get

$$\Gamma(1) = \int_0^\infty e^{-t}\, t^{1-1}\, dt$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt$$

$$(1) = \left[\frac{e^{-t}}{-1}\right] = 1$$

Now, using theorem (2.2.1) for n=2 we have

$$(2) = \Gamma(1 + 1) = 1\ \Gamma(1) = 1.1 = 1$$

For n = 3,

$$(3) = \Gamma(2 + 1) = 2.\ (2) = 2.1 = 2 = 2!$$

Similarly n = 4,

$$(4) = \Gamma(3 + 1) = 3.\Gamma(3) = 3.\ 2! = 3!$$

Hence in general $(n + 1) = n!$.

Theorem (2.3.3): Gamma function is contineous at all positive real numbers.

Proof: Using Weierstrass M- test we get result.

Graph of gamma function for real values $-5 \le x \le 5$ shown in figure 2.3.1

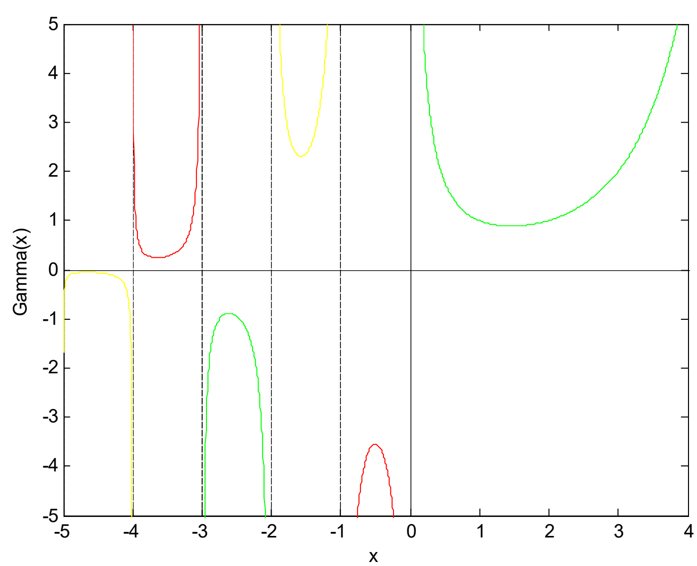

Figure 2.3.1 graph of gamma function

Theorem (2.3.4): If x is any positive real number then prove that

$$\lim_{x \to 0+} \Gamma(x) = \infty.$$

Proof: We know

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$$

$$\Gamma(x) > \int_0^1 e^{-t} t^{x-1}\, dt > \frac{1}{e}\int_0^1 t^{x-1}\, dt$$

The integral $\int_0^1 t^{x-1}\, dt$ is an improper integral and can be evaluated as

$$\int_0^1 t^{x-1}\, dt = \lim_{\delta \to 0+} \int_\delta^1 t^{x-1}\, dt$$

$$= \lim_{\delta \to 0+} \left[\frac{1}{x} - \frac{\delta^x}{x}\right] = \frac{1}{x}$$

Therefore $\Gamma(x) > \frac{1}{\delta x}$, for x > 0 and

Hence $\lim_{x \to 0+} \Gamma(x) = \infty$

Theorem (2.3.5): Prove that, $\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$

Proof: We will prove this by considering the region as shown in figure 2.3.2

From figure required region is given by set as

$$S = \{(x, y) \mid 0 \le x \le R, 0 \le y \le R\}$$

And consider $C_1 = \{ (x, y) \mid x \ge 0, y \ge 0, x^2 + y^2 \le R^2\}$

$$C_2 = \{ (x, y) \mid x \ge 0, y \ge 0, x^2 + y^2 \le 2R^2\}$$

It is obvious that $C_1 \subset S \subset C_2$

Therefore $\iint_{C_1} e^{-x^2-y^2} dA < \iint_S e^{-x^2-y^2} dA < \iint_{C_2} e^{-x^2-y^2} dA$

As region of boundry integrals is circular and middle integral is reactangular therefor by polar coordinates, we have

$$\int_0^{\frac{\pi}{2}} \int_0^R r e^{-r^2}\, dr\, d\theta < \int_0^R \int_0^R e^{-x^2} e^{-y^2} dx\, dy < \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}R} r e^{-r^2}\, dr\, d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1-e^{-R^2}}{2}\, d\theta < \left(\int_0^R e^{-x^2}\, dx\right)\left(\int_0^R e^{-y^2}\, dy\right) < \int_0^{\frac{\pi}{2}} \frac{1-e^{-2R^2}}{2}\, d\theta$$

$$\frac{\pi\left(1 - e^{-R^2}\right)}{4} < \left(\int_0^R e^{-x^2}\, dx\right)^2 < \frac{\pi\left(1 - e^{-2R^2}\right)}{4}$$

Letting limit as $R \to \infty$, we get

$$\left(\int_0^\infty e^{-x^2}\ dx\right)^2 = \frac{\pi}{4}$$

Hence $\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$.

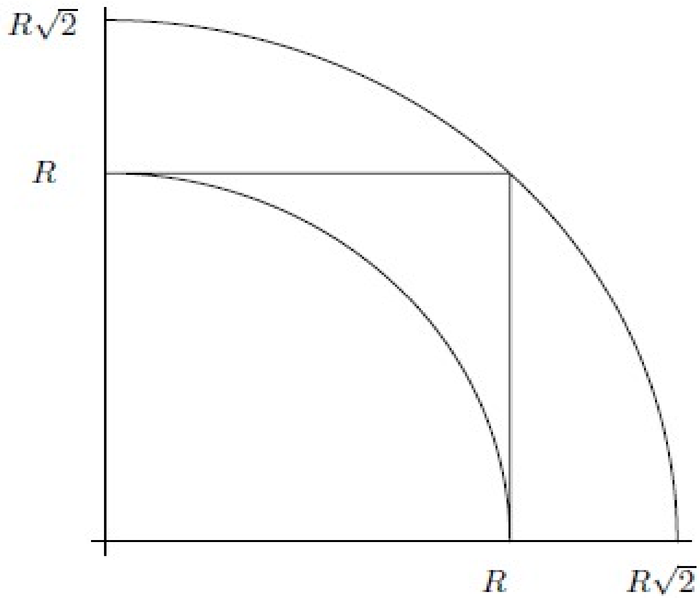

Figure 2.3.2

Theorem (2.3.5): Prove that, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof: put $z = \frac{1}{2}$ in the equation (2.2.1)

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-\frac{1}{2}}\, dt$$

Put $t = u^2$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-u^2}\, u^{-1}\ 2udu$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \text{(by theorem 2.3.4)}$$

Theorem (2.3.6): Prove that, $\Gamma(\,z\,)\Gamma(\,-z\,) = -\frac{\pi}{z\sin(\pi z\,)}$

Proof: $\Gamma(z) = \lim_{n\to\infty}\left\{\frac{n!\ n^z}{z\,(z+1)\ (z+2)\ (z+3)\dots.\ (z+n)}\right\}$ definition (2.3.2)

$$\Gamma(z) = \lim_{n\to\infty}\left(\frac{n^z}{z}\prod_{r=1}^{n}\frac{r}{z+r}\right)$$

Replacing z by -z

$$\Gamma(-z) = \lim_{n\to\infty}\left(\frac{n^{-z}}{-z}\prod_{r=1}^{n}\frac{r}{-z+r}\right)$$

$$\Gamma(z)\Gamma(-z) = \frac{-1}{z^2}\prod_{r=1}^{\infty}\left(1-\frac{z^2}{r^2}\right)^{-1}$$

R.H.S. is the famous weierstrass product, therefore we get

$$\Gamma(z)\Gamma(-z) = \frac{-1}{z^2}\frac{\pi z}{\sin(\pi z)}$$

Hence,

$$(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)} \tag{2.3.4}$$

Theorem (2.3.7): (Reflaction Formula) Prove that,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

Proof: we know by previous theorem

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}$$

$$\Gamma(z)(-z)\Gamma(-z) = \frac{\pi}{\sin(\pi z)}$$

Hence $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ (2.3.5)

Note: In reflection property equation (2.3.5) if we put $z = \frac{1}{2}$, we have

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin\left(\pi\frac{1}{2}\right)}$$

$$(\Gamma(z))^2 = \pi$$

Hence $\Gamma(z) = \sqrt{\pi}$

Some standard values of gamma function are given in table 2.3.1

x	$\Gamma(x)$	x	$\Gamma(x)$
$\frac{1}{2}$	$\sqrt{\pi}$	$-\frac{1}{2}$	$-2\sqrt{\pi}$
$\frac{3}{2}$	$\frac{\sqrt{\pi}}{2}$	$-\frac{3}{2}$	$\frac{4\sqrt{\pi}}{3}$
$\left(\frac{1}{2}+n\right)$	$\frac{(2n)!\sqrt{\pi}}{4^n n!}$	$\left(\frac{1}{2}-n\right)$	$\frac{(-4)^n n!\sqrt{\pi}}{(2n)!}$

Table 2.3.1: Some standard values of gamma function

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Definition of some other functions:

I) Incomplete gamma function:

Gamma function can be written as

$$\Gamma(z) = \gamma(z,x) + \Gamma(z,x)$$

Where $\gamma(z,x)$ is called incomplete gamma function and $\Gamma(z,x)$ is called its complement (Prym's function) and are defined as

$$\gamma(z,x) = \int_0^x e^{-t}\, t^{x-1}\, dt \qquad x > 0$$

and

$$\Gamma(z,x) = \int_x^\infty e^{-t}\, t^{x-1}\, dt \qquad x > 0$$

Special values of incomplete gamma function and its complement for integers z = n ϵ N

$$\gamma(1+n, x) = n!\left[1 - e^{-x} \sum_{k=0}^{n} \frac{x^k}{k!}\right] \qquad n= 0, 1, 2, \ldots..$$

$$\Gamma(1+n, x) = n!\, e^{-x} \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad n= 0, 1, 2, \ldots\ldots$$

II) Reciprocal of gamma function:

The reciprocal Gamma function occurs in many formulas which is defined for all complex numbers z its definition is given in equation (2.3.6) and also graph for its real values is shown in figure 2.3.3.

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \left\{ \frac{z\,(z+1)\,(z+2)\,(z+3)\ldots\,(z+n)}{n!\,n^z} \right\} \qquad (2.3.6)$$

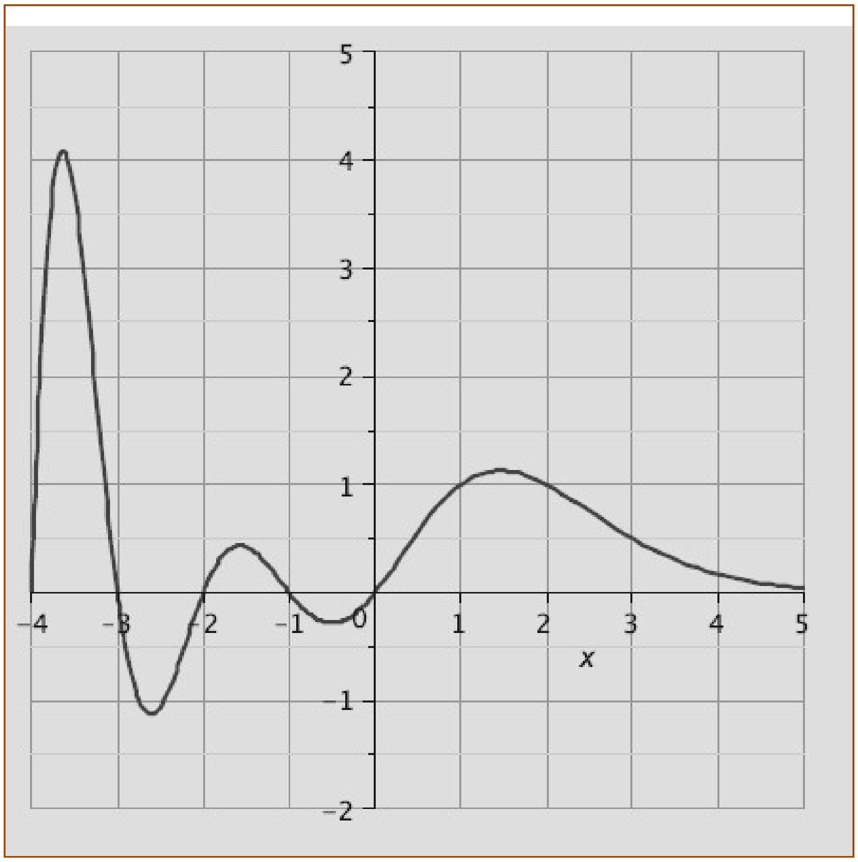

Figure 2.3.3 Reciprocal of gamma function

III) Error Function:

The error function of x is denoted by erf(x), x may be real or complex defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

Some special values:

1) erf(0) = 0

2) erf(-x) = -erf(x)

3) erf(x) + erf_c(x) = 1

4) erf(∞) = 1

IV) Complementry Error Function:

The complementry error function of x is denoted by erf_c(x), x may be real or complex, figure 2.2.2 is the graph of function and is defined as

$$\mathrm{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt$$

Some special values:

1) $\mathrm{erf}_c(0) = 1$

5) $\int_0^\infty \mathrm{erf}_c(x)dx = \frac{1}{\sqrt{\pi}}$

2) $\mathrm{erf}_c(-\infty) = 2$

6) $\int_0^\infty \mathrm{erf}_c{}^2(x)dx = \frac{2-\sqrt{2}}{\sqrt{\pi}}$

3) $\mathrm{erf}_c(+\infty) = 0$

7) $\mathrm{erf}(x) + \mathrm{erf}_c(x) = 1$

4) $\mathrm{erf}_c(-x) = 2 - \mathrm{erf}_c(x)$

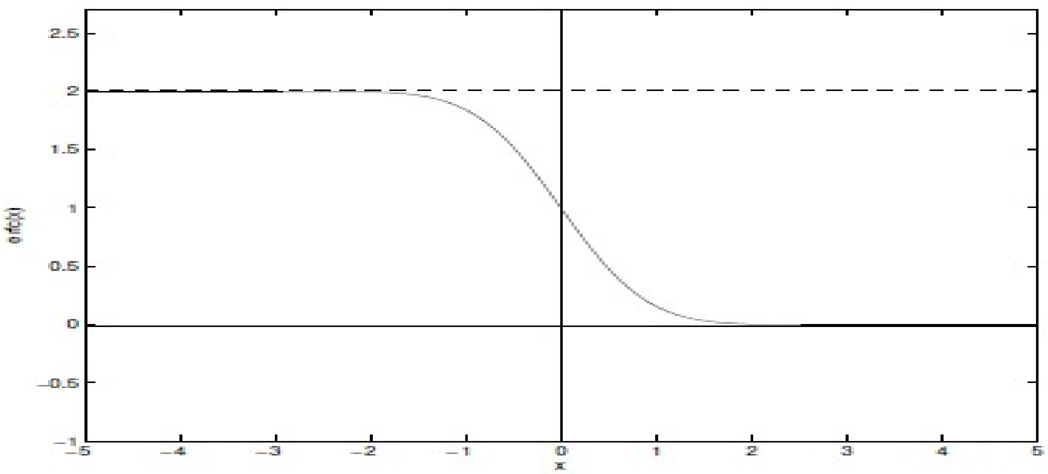

Figure 2.2.2: Graph of complementry error function

2.4 Beta Function:

Beta function is a two variable function which is related with gamma function and widely useful in different field, mostly used in solving definite integral, Euler's integral of first kind i.e. Beta function is defined as

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, \qquad Re(x), Re(y) > 0 \tag{2.4.1}$$

Properties of Beta function:

Theorem (2.4.1): Prove that, $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

Proof: Consider

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u}\, u^{x-1}\, du \int_0^\infty e^{-v}\, v^{y-1}\, dv$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty u^{x-1}\, v^{y-1} e^{-u-v}\, du\, dv$$

Let $t = u + v$, $v = t - u$

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^t u^{x-1} (t-u)^{y-1} e^{-t} \, du \, dt$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^t u^{x-1} (1-\tfrac{u}{t})^{y-1} t^{y-1} e^{-t} \, du \, dt$$

Put $\frac{u}{t} = s$, u = ts means du = t ds

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^1 (ts)^{x-1} (1-s)^{y-1} t^{y-1} e^{-t} t \, ds \, dt$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x+y-1} e^{-t} \, dt \int_0^1 s^{x-1} (1-s)^{y-1} \, ds$$

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\beta(x,y)$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad . \qquad (2.4.2)$$

Theorem (2.4.2): Prove that beta function is symmetric i.e. prove that

β (x, y) = β (y, x)

Proof: $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$

$$\beta(x,y) = \int_1^0 (1-s)^{x-1} s^{y-1} (-ds)$$

$$\beta(x,y) = \int_0^1 s^{y-1}(1-s)^{x-1} \, ds$$

$$\beta(x,y) = \beta(y,x). \qquad (2.4.3)$$

Theorem (2.4.3): Prove that

$$\beta(x, y) = \int_0^\infty \frac{s^{x-1}}{(s+1)^{x+y}} \, ds , \quad \Re(x) > 0, \Re(y) > 0.$$

Proof: $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$

Substitute $t = \frac{s}{(s+1)}$

$$\beta(x,y) = \int_0^\infty s^{x-1}(s+1)^{-x+1}(s+1)^{-y+1}(s+1)^{-2} \, ds$$

$$\beta(x, y) = \int_0^\infty \frac{s^{x-1}}{(s+1)^{x+y}} \, ds , \quad \Re(x) > 0, \Re(y) > 0. \qquad (2.4.4)$$

Theorem (2.4.4): Prove that

$$\beta(x,y) = 2\int_0^{\frac{\pi}{2}} (cos\theta)^{2x-1} (sin\theta)^{2y-1} d\theta, \quad x > 0, y > 0.$$

Proof: Consider

$$\Gamma(x)\Gamma(y) = \int_0^\infty e^{-u} u^{x-1} \, du \int_0^\infty e^{-v} v^{y-1} \, dv$$

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty u^{x-1} v^{y-1} e^{-u-v} \, du \, dv$$

Substitute u= s^2 and v=t^2

$$\Gamma(x)\Gamma(y) = \int_0^\infty \int_0^\infty (s^2)^{x-1} (t^2)^{y-1} e^{-s^2-t^2} 2s2tdsdt$$

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \int_0^\infty s^{2x-1} t^{2y-1} e^{-s^2-t^2} dsdt$$

On changing to polar coordinate

$$\Gamma(x)\Gamma(y) = 4\int_0^{\frac{\pi}{2}} \int_0^\infty (r\cos\theta)^{2x-1} (r\sin\theta)^{2y-1} e^{-r^2} rdrd\theta$$

$$\Gamma(x)\Gamma(y) = 4\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \int_0^\infty r^{2x+2y-1} e^{-r^2} dr$$

$$\Gamma(x)\Gamma(y) = 4\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \frac{\Gamma(x+y)}{2}$$

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\sin\theta)^{2y-1} d\theta \qquad (2.4.5)$$

Theorem (2.4.3): Prove that

$$\int_a^b (s-a)^{x-1}(b-s)^{y-1}ds = (b-a)^{x+y-1}\beta(x, y) ,$$

$$\Re(x) > 0, \Re(y) > 0.$$

Proof: $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$

Substitute $t = \frac{(s-a)}{(b-a)}$

$$\beta(x,y) = \int_a^b (s-a)^{x-1}(b-a)^{-x+1}(b-s)^{y-1}(b-a)^{-y+1}(b-a)^{-1}\, ds$$

$$\beta(x,y) = (b-a)^{-x-y+1} \int_0^1 (s-a)^{x-1}(b-s)^{y-1}\, ds$$

Hence $\int_a^b (s-a)^{x-1}(b-s)^{y-1}ds = (b-a)^{x+y-1}\beta(x, y)$,

$$\Re(x) > 0, \Re(y) > 0.$$

Some standard values of Beta function:

x, y	β (x, y)	x, y	β (x, y)	
$\frac{1}{2}, \frac{1}{2}$	π	$x, 1$	$\frac{1}{x}$	
$\frac{1}{3}, \frac{1}{3}$	$\frac{2\pi}{\sqrt{3}}$	$x, 1-x$	$\frac{\pi}{\sin(\pi x)}$	
$\frac{1}{4}, \frac{3}{4}$	$\pi\sqrt{2}$	m, n	$\frac{(m-1)!(n-1)!}{(m+n-1)!}$	$m, n \geq 1$

Table 2.4.1: Special Values of Beta function

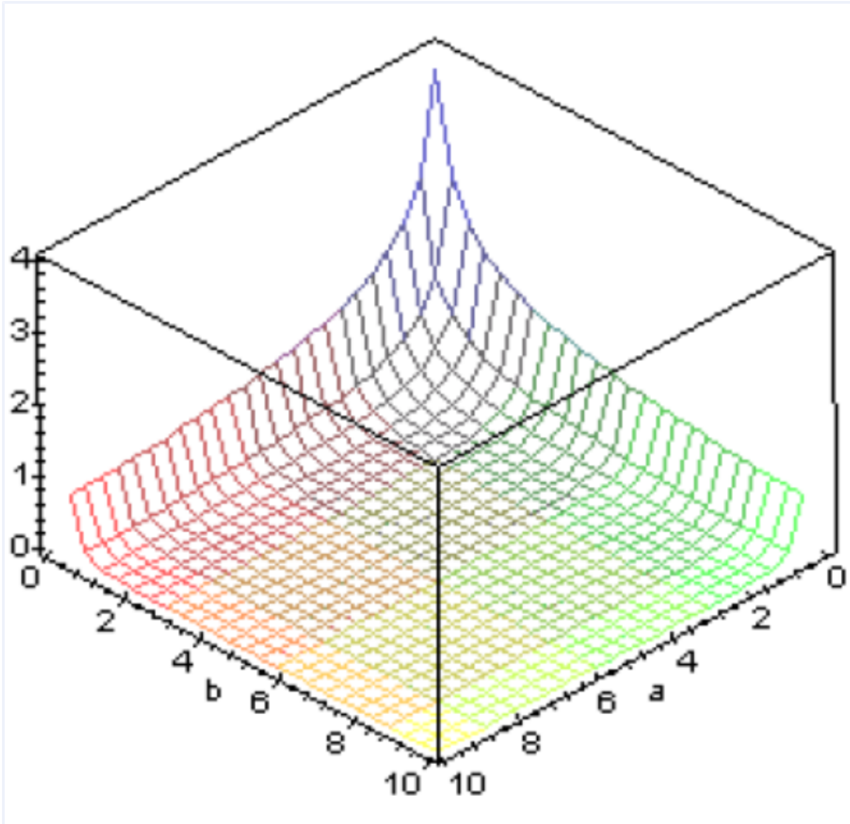

Figure 2.4.1: Graph of Beta function

Incomplete Beta function:

Incomplete beta function is defined as

$$\beta_\alpha(x, y) = \int_0^\alpha t^{x-1}(1-t)^{y-1}\, dt \quad 0 \le \alpha \le 1 \tag{2.4.5}$$

Normalized or regularized form of incomplete beta function is defined as

$$I_\alpha(x, y) = \frac{\beta_\alpha(x,y)}{\beta(x,y)} \tag{2.4.6}$$

2.5 Mittag-Leffler function (MLF):

Mittag-Leffler function is the generalization of exponential function, which plays very important role in fractional calculus and fractional modeling. It is important to note that the role of the Mittag-Leffler function as the queen function of fractional calculus. The importance and popularity of the Mittag-Leffler function was found when its uses to fractional calculus and its applications were fully understood. Different aspects of this function in fractional theory and its applications/ modeling in fractional calculus have been described. One parameter Mittag-Leffler function was introduced by Gösta Magnus Mittag-Leffler as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha, z \in C, \quad Re(\alpha) > 0. \qquad (2.5.1)$$

Two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}, \quad \alpha, \beta, z \in C, \quad Re(\alpha), Re(\beta) > 0. \qquad (2.5.2)$$

Three parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_n z^k}{n!\,\Gamma(\beta+\alpha k)}, \quad \alpha, \beta, z \in C, \quad Re(\alpha), Re(\beta), \gamma > 0 \qquad (2.5.3)$$

where $(\gamma)_n = \begin{cases} \gamma(\gamma+1)(\gamma+2)\ldots\ldots(\gamma+n-1), & for\ n \geq 1 \\ 1, & for\ n = 1 \end{cases}$

$= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, called Pocchamber symbol.

Equation (2.5.2) is the generalization of equation (2.5.1) which is introduced and was studied by A. Wiman in 1905, also studied by Humbert and R.P.Agrawal in 1953. For $\beta = 1$ MLF of two parameter is reduces to MLF of one parameter. Equation (2.5.3) is an again extension to Mittag-Leffler function of one and two parameter which is introduced by Prabhakar, when γ =1 MLF of three parameter reduces to MLF of two parameter and when $\gamma = 1$, $\beta = 1$ MLF of three parameter reduces to MLF of one parameter.

Three parameter Mittag-Leffler function of another type also introduced by Kilbas and Saigo, to solve particular type of fractional differential equation. Both these three parameter form of Mittag Leffler function are used as an explicit representation of solutions to fractional differential equations and fractional integral equations. Shukla and Prajapati was given further generalization of Prabhakar's function (2.5.3) and Saxena and Nishimoto was combined with the definition given by Shukla and Prajapati with the definition of three parameter given by Kilbas and Saigo.

Further generalizations of the Mittag-Leffler function (multi-parametric Mittag-Leffler functions) were proposed, some of such generalizations was discussed in. Mittag Leffler functions of one, two and three parameter are the basic versions and are used to generate multi-parametric Mittag Leffler functions. Fractional operators and the Mittag-Leffler function are widely used in many sections of science and engineering and other applied sciences. In the last one to two decades this function has come into prominence, due to the huge potential of its applications in solving the problems of biological, physical engineering, chemical, economical and earth sciences etc.

Some special values of one parameter MLF $E_\alpha(z)$:

1) $$E_0(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+0k)} = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \quad (2.5.4)$$

2) $$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z \quad (2.5.5)$$

3) $$E_2(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+2k)} = \cosh(\sqrt{z}), \quad z \in C \quad (2.5.6)$$

4) $$E_3(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+3k)} = \frac{1}{2}\left[e^{z^{\frac{1}{3}}} + 2e^{-\frac{1}{2}z^{\frac{1}{3}}} \cos\left(\frac{\sqrt{3}}{2} z^{\frac{1}{3}}\right)\right] \quad (2.5.7)$$

Properties of Mittag Leffler function:

Theorem (2.5.1): Prove that

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)} \quad (2.5.8)$$

Proof: Consider

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}$$

$$E_{\alpha,\beta}(z) = \sum_{k=-1}^{\infty} \frac{z^{k+1}}{\Gamma(\alpha+\beta+\alpha k)}$$

$$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + \sum_{k=0}^{\infty} \frac{z^{k+1}}{\Gamma(\alpha+\beta+\alpha k)}$$

$$E_{\alpha,\beta}(z) = zE_{\alpha,\alpha+\beta}(z) + \frac{1}{\Gamma(\beta)}, \qquad R(\beta) > 0.$$

Theorem (2.5.2): Prove that

$$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z) \quad (2.5.9)$$

Proof: Consider R.H.S. of equation (2.5.5)

$$\text{R.H.S.} = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z)$$

$$= \beta \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+1+\alpha k)} + \alpha z \frac{d}{dz} \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+1+\alpha k)}$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha k+\beta) z^k}{\Gamma(\beta+1+\alpha k)}$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta+\alpha k)}$$

$$= E_{\alpha,\beta}(z) = L.H.S$$

Theorem (2.5.3): Prove that

$$\left(\frac{d}{dz}\right)^r \left[z^{\beta-1}E_{\alpha,\beta}(z^{\alpha})\right] = z^{\beta-r-1}E_{\alpha,\beta-r}(z^{\alpha}), \quad (2.5.10)$$

where $R(\beta - r) > 0$, $r = 0,1,2,\ldots$.

Proof: Consider L.H.S. of equation (2.5.10)

$$L.H.S. = \left(\frac{d}{dz}\right)^r \left[z^{\beta-1}E_{\alpha,\beta}(z^{\alpha})\right]$$

$$= \left(\frac{d}{dz}\right)^r \sum_{k=0}^{\infty} \frac{z^{\alpha k+\beta-1}}{\Gamma(\beta+\alpha k)}$$

$$= \sum_{k=0}^{\infty} \frac{z^{\alpha k+\beta-r-1}}{\Gamma(\beta-r+\alpha k)}, \quad R(\beta - r) > 0$$

$$= z^{\beta-r-1}E_{\alpha,\beta-r}(z^{\alpha})$$

= R.H.S.

Theorem (2.5.4): Prove that

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2} \frac{z^k}{\Gamma(k+1)}\right\}, \quad m=1, 2, 3.. \quad (2.5.11)$$

Proof: Consider first m=1

$$E_{1,1}(z) = e^z$$

For m = 2, $E_{1,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)}$

$$E_{1,2}(z) = \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{\Gamma(k+2)}$$

$$E_{1,2}(z) = \frac{1}{z}\left(e^z - 1\right)$$

For m = 3, $E_{1,3}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)}$

$$E_{1,3}(z) = \frac{1}{z^2} \sum_{k=0}^{\infty} \frac{z^{k+2}}{\Gamma(k+3)}$$

$$E_{1,3}(z) = \frac{1}{z^2}\left(e^z - z - 1\right)$$

Hence for any m = 1, 2, 3…

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left\{e^z - \sum_{k=0}^{m-2} \frac{z^k}{\Gamma(k+1)}\right\}.$$

Some Special values of Mittag-Leffler function are given in table 3.5.1:

α	$E_{\alpha}(z)$	α, β	$E_{\alpha,\beta}(z)$

0	$\frac{1}{(1-z)}$, |z| < 1	2, 1	$\cosh(z)$
1	e^z	2, 2	$\frac{\sinh(z)}{z}$
2	$\cosh(\sqrt{z})$	1/2, 1	$e^{z^2}\,\mathrm{erf}_c(-z)$

Table 3.5.1: Special Values of Mittag-Leffler function

2.6 The Mellin-Ross Function:

The Mellin-Ross function $E_t(\alpha, x)$ closely related to Mittag-Leffler function, incomplete gamma etc it supports in finding fractional integral of exponential functions and related functions. It is definied as

$$E_t(\alpha, x) = t^{\alpha} e^{xt} \gamma(\alpha, t) \qquad (2.6.1)$$

Where $\gamma(\alpha, t)$ is incomplete gamma function. It can also be written as

$$E_t(\alpha, x) = t^{\alpha} \sum_{k=0}^{\infty} \frac{(xt)^k}{\Gamma(k+\alpha+1)} = t^{\alpha} E_{1,\alpha+1}(xt) \qquad (2.6.2)$$

2.7 Conclusion:

Special functions are the backbons of fractional calculus, without special function there should not be the birth of fractional calculus. Whateve may be the development found in fractional calculus its only due to special functions. Gamma function is the son of fractional calculus and Mittag-Leffler function is the life partner of fractional calculus. Growth in development of special functions means growth in developments of fractional calculus.

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CHAPTER -III

DIFFERENT APPROACHES OF FRACTIONAL CALCULUS

3.1 Fractional Calculus:

Fractional calculus was for the most part a study held in reserve for the best minds in mathematics except some few mathematicians many are dabbled with fractional calculus and the mathematical consequences. Many are using their own definition, notation, concept and methodology. The most popularized of these definitions are the Grünwald-Letnikov, Riemann-Liouville and Caputo definition. Numerous applications of fractional calculus have been found. However, these applications and the mathematical background surrounding fractional calculus are far from paradoxical, while the physical meaning is very difficult (possibly impossible) to grasp. Here we study development and properties of some famous approaches of fractional calculus.

3.2 Grünwald-Letnikov Definition:

Anton Karl Grünwald proposed the Grünwald definition of differintegrals in 1867 at Prague. Same type of definition was also given by Aleksey Vasilievich Letnikov in 1868 at Moscow. Hence this definition is sometimes known as the Grünwald -Letnikov definition [3,5].

Grünwald –Letnikov approach achieved by successive differentiation. It begins by first order derivative of function y=f(x), given by

$$\frac{dy}{dx} = D^1 f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$D^2 f(x) = \lim_{h\to 0} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2}$$

$$D^3 f(x) = \lim_{h\to 0} \frac{f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)}{h^3}$$

$$D^4 f(x) = \lim_{h\to 0} \frac{f(x+4h) - 6f(x+3h) + 4f(x+2h) - 6f(x+h) + f(x)}{h^4}$$

Continuing the successive differentiation n^{th} order derivative can be written as

$$D^n f(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(x - rh) \qquad (3.2.1)$$

Where $\binom{n}{r}$ is binomial coefficient and

$$\binom{n}{r} = \frac{\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)}$$

To generalize this expression for non-integer values of $n = \alpha \epsilon$ R, we get understood with the binomial coefficient using the Gamma Function in place of the standard factorial. But, for the upper limit of the summation n goes to infinity as $\frac{x-a}{h}$ with $a < x$ (where x and a are the upper and lower limits of differentiation, respectively. With these changes equation for Grünwald – Letnikov fractional derivative becomes

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^n} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \binom{\alpha}{r} f(x - rh)$$

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \frac{\Gamma(\alpha+1)}{\Gamma(r+1)\Gamma(\alpha-r+1)} f(x - rh)$$

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \frac{\Gamma(\alpha+1)}{r!\,\Gamma(\alpha-r+1)} f(x - rh) \qquad (3.2.2)$$

Equation (3.7.2) represents definition of Grünwald–Letnikov for fractional differentiation.

To get Grünwald–Letnikov fractional integral approach, alter GL definition of fractional differentiation with negative α. For negative α binomial coefficient in equation (3.7.1) becomes

$$\binom{-n}{r} = \frac{-n(-n-1)(-n-2)(-n-3)\ldots\ldots(-n-r+1)}{r!}$$

$$\binom{-n}{r} = (-1)^r \frac{n(n+1)(n+2)(n+3)\ldots\ldots(n+r-1)}{r!}$$

$$\binom{-n}{r} = (-1)^r \frac{(n+r-1)!}{(n-1)!\,r!}$$

Using Gamma function this factorial expression can be written for n =α ϵ R as

$$\binom{-\alpha}{r} = (-1)^r \frac{(\alpha+r-1)!}{(\alpha-1)!\,r!} = (-1)^r \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)\,r!}$$

Therefore fractional integral of Grünwald–Letnikov becomes

$$D^{-\alpha} f(x) = \lim_{h \to 0} h^{-\alpha} \sum_{r=0}^{\frac{x-a}{h}} \frac{\Gamma(\alpha+r)}{r!\,\Gamma(\alpha)} f(x - rh) \qquad (3.2.3)$$

Limits vanishes with integer order derivatives therefore in fractional derivatives limits must be considered This also means that fractional derivatives are nonlocal, which may be the reason that makes this kind of derivatives less useful in describing nature.

Examples:

I) Evaluate $D^{\alpha} f(x)$, for α = ½ and f(x) = 1, a constant.

$$D^{\alpha} f(x) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \frac{\Gamma(\alpha+1)}{r!\,\Gamma(\alpha-r+1)} f(x - rh)$$

From the results in (page no. 20 [21]), we have

$D^{\alpha} f(x) = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}$,

$D^{\frac{1}{2}} 1 = \frac{x^{-\frac{1}{2}}}{\Gamma(1-\frac{1}{2})} = \frac{1}{\sqrt{\pi x}}$

3.3 Riemann-Liouville Definition:

The most commonly used definition of a fractional differintegral was proposed by Riemann and Liouville. The Riemann-Liouville definition allows for the calculation of a differintegral of any real order. Like Grünwald–Letnikov approach to fractional calculus begins with successive differentiation, Riemann-Liouville approach to fractional calculus begins with n-fold integrals. For evaluating such n-fold integrals, Cauchy's integral formula was used. Consider an integral

$$D^{-1}f(x) = \int_0^x f(r_1)\, dr_1$$

$$D^{-2}f(x) = \int_0^x \int_0^{r_2} f(r_1)\, dr_1 dr_2$$

$$= \int_0^x \int_{r_1}^x f(r_1)\, dr_2 dr_1$$

$$= \int_0^x f(r_1) \int_{r_1}^x dr_2 dr_1$$

$$= \int_0^x f(r_1)(x - r_1) dr_1$$

Similarly applying procedure

$$D^{-n}f(x) = \frac{1}{(n-1)!} \int_0^x f(r_1)(x - r_1)^{n-1} dr_1$$

In general

$$D^{-n}f(x) = \frac{1}{(n-1)!} \int_0^x f(r)(x - r)^{n-1} dr \qquad (3.3.1)$$

Hence its generalization to arbitrary (non-integer) values is

$$D^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(r)}{(x-r)^{\alpha+1}} dr \qquad (3.3.2)$$

In limit form it can be written as

$$_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(r)}{(x-r)^{\alpha+1}} dr \qquad (3.3.3)$$

This form is not valid if the real part of α is positive or zero since the integral diverges and in this case it can be used for evaluating generalized integrals. Equation (2.8.3) to become valid using ordinary derivative, we may write it as

$$_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x - r)^{n-\alpha-1} f(r)\, dr \qquad (3.3.4)$$

Riemann-Liouville Fractional integral:

Thus in general Riemann-Liouville Fractional integral of order α > 0 of the function f(x) is defined as

$$I^{\alpha}f(x) = D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1}f(r)\,dr \qquad (3.3.5)$$

Left hand and Right hand Riemann-Liouville Fractional integral are defined as

$$I_{a+}^{\alpha}f(x) = D_{a+}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1}f(r)\,dr \qquad (3.3.6)$$

$$I_{b-}^{\alpha}f(x) = D_{b-}^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-r)^{\alpha-1}f(r)\,dr \qquad (3.3.7)$$

Examples:

I) Evaluate $I^{\alpha}x^{\mu}$, where $\mathcal{R}(\alpha) > 0, \mu > -1$.

$$I^{\alpha}x^{\mu} = \frac{1}{\Gamma(\alpha)}\int_0^x (x-r)^{\alpha-1}r^{\mu}\,dr$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^x \left(1-\frac{r}{x}\right)^{\alpha-1} x^{\alpha-1}r^{\mu}\,dr$$

On substituting $\frac{r}{x} = t$

$$= \frac{1}{\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1}x^{\alpha-1}(xt)^{\mu}\,xdt$$

$$= \frac{1}{\Gamma(\alpha)}x^{\alpha+\mu}\int_0^1 (1-t)^{\alpha-1}t^{\mu}\,dt$$

$$= \frac{1}{\Gamma(\alpha)}x^{\alpha+\mu}\,B(\alpha,\mu+1)$$

$$I^{\alpha}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu} \qquad (3.3.5.1)$$

Note: For the result (2.3.5.1) we see some particular examples

i) For α = α and $x^{\mu} = C$, where C is constant

$$I^{\alpha}c = c\,\frac{\Gamma(1)}{\Gamma(\alpha+1)}x^{\alpha} = \frac{c}{\Gamma(\alpha+1)}x^{\alpha},$$ where C is a constant.

ii) For α = 1/2 and $x^{\mu} = x^0$

$$I^{\alpha}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu} \Rightarrow I^{\frac{1}{2}}x^0 = \frac{\Gamma(1)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$$

iii) For α = 3/2 and $x^{\mu} = x^0$

$$I^{\alpha}x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu} \Rightarrow I^{\frac{3}{2}}x^0 = \frac{\Gamma(1)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{1}{2}} = \frac{4}{3}\sqrt{\frac{x}{\pi}}$$

iv) For α = 1/2 and $x^{\mu} = x^1$

$$I^{\alpha}\,x^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu}\Rightarrow I^{\frac{1}{2}}\,x^{1}=\frac{\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}}=\frac{4}{3}\sqrt{\frac{x^{3}}{\pi}}$$

v) For $\alpha = 3/2$ and $x^{\mu}=x^{1}$

$$I^{\alpha}\,x^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu}\Rightarrow I^{\frac{3}{2}}\,x^{1}=\frac{\Gamma(2)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}=\frac{8}{15}\sqrt{\frac{x^{5}}{\pi}}$$

vi) For $\alpha = 1/2$ and $x^{\mu}=x^{2}$

$$I^{\alpha}\,x^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu}\Rightarrow I^{\frac{1}{2}}\,x^{2}=\frac{\Gamma(3)}{\Gamma\left(\frac{7}{2}\right)}x^{\frac{5}{2}}=\frac{16}{15}\sqrt{\frac{x^{5}}{\pi}}$$

vii) For $\alpha = 3/2$ and $x^{\mu}=x^{2}$

viii) $$I^{\alpha}\,x^{\mu}=\frac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}x^{\alpha+\mu}\Rightarrow I^{\frac{3}{2}}\,x^{2}=\frac{\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}x^{\frac{7}{2}}=\frac{32}{105}\sqrt{\frac{x^{7}}{\pi}}$$

From above examples one may take for granted that it is easy to assess fractional integral, but which is not fact since evaluating fractional integrals of some other elementary basic functions (such as trigonometric, exponential, logarithmic etc.) results in higher transcendental functions then what about others ?

II) Evaluate $I^{\alpha}\,e^{ax}$, where a is constant, $\mathcal{R}(\alpha) > 0, \mu > -1$.

By definition

$$I^{\alpha}\,e^{ax}=\frac{1}{\Gamma(\alpha)}\int_0^x (x-r)^{\alpha-1}e^{ar}\,dr \qquad (3.3.5.2)$$

Let y = x – r then

$$I^{\alpha}\,e^{ax}=-\frac{e^{ax}}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}e^{-ay}\,dy$$

This is not an elementary function therefore we need some special function having solution to such integrals. This can be evaluated by Millen-Ross function as

$$I^{\alpha}\,e^{ax}=E_x(\propto,a)=x^{\propto}E_{1,\propto+1}(ax)$$

III) Evaluate $I^{\alpha}\cos(ax)$, where a is constant, $\mathcal{R}(\alpha) > 0, \mu > -1$.

$$I^{\alpha}\,\cos(ax)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-r)^{\alpha-1}\cos(ar)\,dr$$

Let y = x – r then

$$I^{\alpha}\,\cos(ax)=-\frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\cos(a(x-y))\,dy$$

This is not an elementary function.

IV) Evaluate $I^{\alpha}\sin(ax)$, where a is constant, $\mathcal{R}(\alpha) > 0, \mu > -1.$

$$I^{\alpha}\sin(ax) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-r)^{\alpha-1}\sin(ar)\ dr$$

Let y = x – r then

$$I^{\alpha}\sin(ax) = -\frac{1}{\Gamma(\alpha)}\int_0^x y^{\alpha-1}\sin(a(x-y))\ dy$$

This is not an elementary function.

Properties of Riemann-Liouville Fractional integral:

Theorem 3.3.1: Suppose that α > 0, x > 0, α, a, x ϵ R then Riemann-Liouville Fractional integral $I^{\alpha}f(x)$ for α = 0 satisfies $I^{0}f(x) = f(x)$

Proof: It is by convention $I^0 = I$, identity function

$$I^0 f(x) = If(x) = f(x)$$

Theorem 3.3.2: Suppose that α > 0, x > 0, α, a, x ϵ R then Riemann-Liouville Fractional integral $I^{\alpha}f(x)$ satisfies linearity i.e.

$$I^{\alpha}\{\lambda f(x) + \mu g(x)\} = \lambda I^{\alpha}f(x) + \mu I^{\alpha}g(x), \qquad \lambda, \mu \in C \qquad (3.3.8)$$

Proof: Consider

$$L.H.S. = I^{\alpha}\{\lambda f(x) + \mu g(x)\}$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1}\big(\lambda f(r) + \mu g(r)\big)dr$$

$$= \frac{1}{\Gamma(\alpha)}\int_a^x \{\big(\lambda(x-r)^{\alpha-1}f(r) + \mu(x-r)^{\alpha-1}g(r)\big)\}dr$$

$$= \frac{\lambda}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1}f(r)\ dr + \frac{\mu}{\Gamma(\alpha)}\int_a^x (x-r)^{\alpha-1}g(r)\ dr$$

$$= \lambda I^{\alpha}f(x) + \mu I^{\alpha}g(x) = R.H.S.$$

Theorem 3.3.3: Suppose that α > 0, β > 0, x > 0, α, x ϵ R f(x) is continuous for x ≥ 0 then Riemann-Liouville Fractional integral satisfies exponent law i.e. following equality holds

$$I^{\alpha}\left(I^{\beta}f(x)\right) = I^{\beta}\left(I^{\alpha}f(x)\right) = I^{\alpha+\beta}f(x) \qquad (3.3.9)$$

Proof: consider by definition

$$I^{\alpha}\left(I^{\beta}f(x)\right) = \frac{1}{\Gamma(\alpha)}\int_0^y (y-r)^{\alpha-1}\left(I^{\beta}f(x)\right)\ dr$$

$$= \frac{1}{\Gamma(\alpha)}\int_0^y (y-r)^{\alpha-1}\left[\frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}f(t)\ dt\right]\ dr$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y (y-r)^{\alpha-1}\left[\int_0^x (x-t)^{\beta-1}f(t)\ dt\right]\ dr$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha,\beta) \int_o^x (x-t)^{\alpha+\beta-1} f(t)\, dt \qquad (3.3.10)$$

Using Dirichlet's formula, where B is the beta function.

Similarly

$$I^{\beta}\left(I^{\alpha} f(x)\right) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} B(\alpha,\beta) \int_o^x (x-t)^{\alpha+\beta-1} f(t)\, dt \qquad (3.3.11)$$

And

$$I^{\alpha+\beta} f(x) = \frac{1}{\Gamma(\alpha+\beta)} \int_a^x (x-r)^{\alpha+\beta-1} f(r)\, dr \qquad (3.3.12)$$

From equations (3.3.10), (3.3.11) and (3.3.12), law of exponent 32.3.9) satisfied by Riemann-Liouville Fractional integral.

Riemann-Liouville Fractional derivative:

In general Riemann-Liouville Fractional derivative of order $\alpha > 0$ and $n-1 < \alpha < n$, $n \epsilon N$ of the function f(x) is as defined in equation (3.8.4) as

$${}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1} f(r)\, dr \qquad (3.3.13)$$

If $0 < \alpha < 1$, we obtain

$$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(r)}{(x-r)^{\alpha}}\, dr$$

Let $\alpha > 0$ and $n-1 < \alpha < n$, $n \epsilon N$, and $a < x < b$, Left hand and Right hand Riemann-Liouville Fractional Derivative is defined as

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt\,, \qquad (3.3.14)$$

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_x^b (x-t)^{n-\alpha-1} f(t)\, dt\,, \qquad (3.3.15)$$

respectively.

Examples:

I) Evaluate $D^{\alpha} x^{\mu}$, where $\mathcal{R}(\alpha) > 0$, $\mu \geq 0$.

Let $0 < \alpha < 1$ therefore $n = 1$

$${}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1} f(r)\, dr$$

$$D^{\alpha} x^{\mu} = \frac{1}{\Gamma(1-\alpha)} \left(\frac{d}{dx}\right)^1 \int_a^x (x-r)^{-\alpha} r^{\mu}\, dr$$

$$D^{\alpha} x^{\mu} = \left(\frac{d}{dx}\right) \left[\frac{1}{\Gamma(1-\alpha)} \int_a^x (x-r)^{-\alpha} r^{\mu}\, dr\right]$$

$$D^{\alpha}\, x^{\mu} = \mathrm{D}[I^{1-\alpha}\, x^{\mu}]$$

$$D^{\alpha}\, x^{\mu} = \mathrm{D}\left[\frac{\Gamma(\mu+1)}{\Gamma((\mu-\alpha+1)+1)}\, x^{\mu-\alpha+1}\right]$$

$$D^{\alpha}\, x^{\mu} = (\mu-\alpha+1)\frac{\Gamma(\mu+1)}{(\mu-\alpha+1)\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha}$$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \qquad (3.3.16)$$

Note: For the result (3.3.16) we see some particular examples

i) For α = α and $x^{\mu} = C$, where C is constant

$$D^{\alpha}\, C = \frac{C}{\Gamma(-\alpha+1)}\, x^{-\alpha} \neq 0 \qquad (3.3.16.1)$$

ii) For α = 1/2 and $x^{\mu} = x^{0}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{1}{2}}\, x^{0} = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)}x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi x}}$$

iii) For α = 3/2 and $x^{\mu} = x^{0}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{3}{2}}\, x^{0} = \frac{\Gamma(1)}{\Gamma\left(-\frac{1}{2}\right)}x^{-\frac{3}{2}}$$

iv) For α = 1/2 and $x^{\mu} = x^{1}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{1}{2}}x^{1} = \frac{\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$$

v) For α = 3/2 and $x^{\mu} = x^{1}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{3}{2}}\, x^{1} = \frac{\Gamma(2)}{\Gamma\left(\frac{1}{2}\right)}x^{\frac{-1}{2}} = \frac{1}{\sqrt{\pi x}}$$

vi) For α = 1/2 and $x^{\mu} = x^{2}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{1}{2}}\, x^{2} = \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}x^{\frac{3}{2}} = \frac{8}{3}\sqrt{\frac{x^{3}}{\pi}}$$

vii) For α = 3/2 and $x^{\mu} = x^{2}$

$$D^{\alpha}\, x^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}\, x^{\mu-\alpha} \Rightarrow D^{\frac{3}{2}}\, x^{2} = \frac{\Gamma(3)}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} = 4\sqrt{\frac{x}{\pi}}$$

Some properties of Riemann-Liouville fractional Derivative:

Theorem 3.3.4: Suppose that x > 0, α, a, x ϵ R then Riemann-Liouville Fractional derivative $D^{\alpha}f(x)$ for α = 0 satisfies

$$D^{0}f(x) = f(x)$$

Proof: It is by convention $D^{0} = I$, identity function

$$D^0 f(x) = If(x) = f(x)$$

Theorem 3.3.5: Suppose that α > 0, x > 0, α, a, x ϵ R, n-1 < α < n, n ϵ N λ, μ ϵ C then Riemann-Liouville Fractional derivative $D^\alpha f(x)$ satisfies linearity i.e.

$$D^\alpha\{\lambda f(x) + \mu g(x)\} = \lambda D^\alpha f(x) + \mu D^\alpha g(x), \qquad (3.3.8)$$

Where λ and μ are constants.

Proof: Consider

$$L.H.S. = D^\alpha\{\lambda f(x) + \mu g(x)\}$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1}\left(\lambda f(r) + \mu g(r)\right) dr$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \left[\left((x-r)^{n-\alpha-1}\lambda f(r)\right) + \left((x-r)^{n-\alpha-1}\mu g(r)\right)\right] dr$$

$$= \frac{\lambda}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1} f(r)\, dr +$$

$$\frac{\mu}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1} g(r)\, dr$$

$$= \lambda D^\alpha f(x) + \mu D^\alpha g(x) = R.H.S.$$

Hence Riemann Liouville fractional derivative satisfies Linearity property.

Theorem 3.3.5: Suppose that α > 0, x > 0, α, a, x ϵ R, n-1 < α < n, n, m ϵ N then Riemann-Liouville Fractional derivative $D^\alpha f(x)$ satisfies

$$D^m\, D^\alpha f(x) = D^{m+\alpha} f(x)$$

Proof: Consider

$$L.H.S. = D^m\, D^\alpha f(x)$$

$$= D^m \left(\frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-r)^{n-\alpha-1} f(r)\, dr\right)$$

$$= D^m \left(\left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-r)^{n-\alpha-1} f(r)\, dr\right)$$

$$= D^m(D^n I^{n-\alpha} f(x))$$

$$= D^{m+n} I^{n-\alpha} f(x)$$

$$= D^{m+\alpha} f(x) = R.H.S.$$

Note: $D^m\, D^\alpha f(x) \neq D^{\alpha+m} f(x)$

Figure (3.8.1) show graph of Riemann-Liouville fractional integral of function $I^\alpha f(x) = D^{-\alpha} f(x)$ where f(x) = y = x for different values of α between 0 and 2.

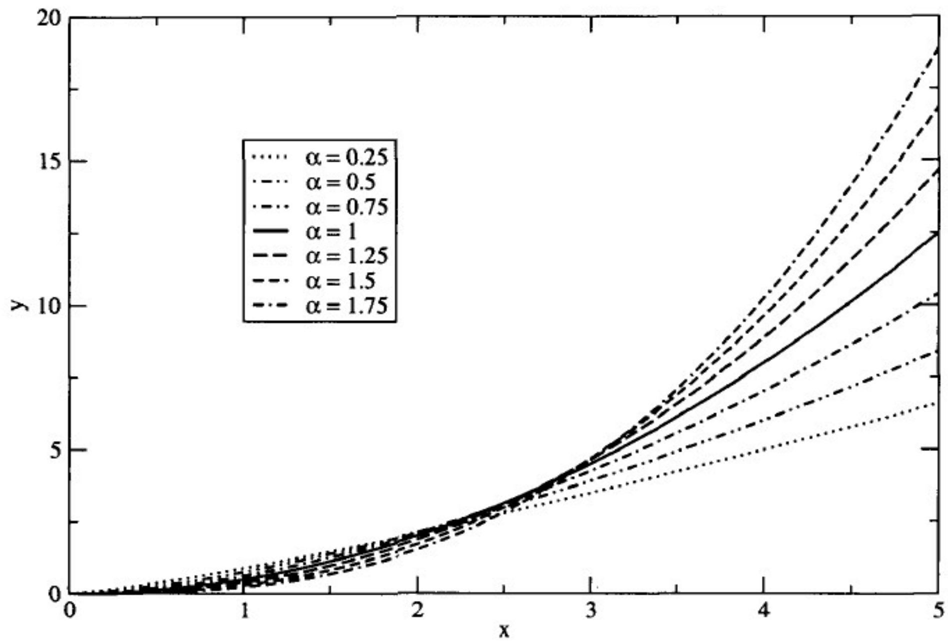

Figure 3.8.1: Riemann-Liouville integral I^{α} of y = x for different values of α.

Figure (3.8.2) show graph of Riemann-Liouville fractional integral of function $I^{\alpha}f(x) = D^{-\alpha}f(x)$ where f(x) = y = e^{-x} for different values of α between 0 and 2. Here we observe that there is a discontinuous position of function at x= 0 between first order integral and arbitrary order between 0 < α < 1.

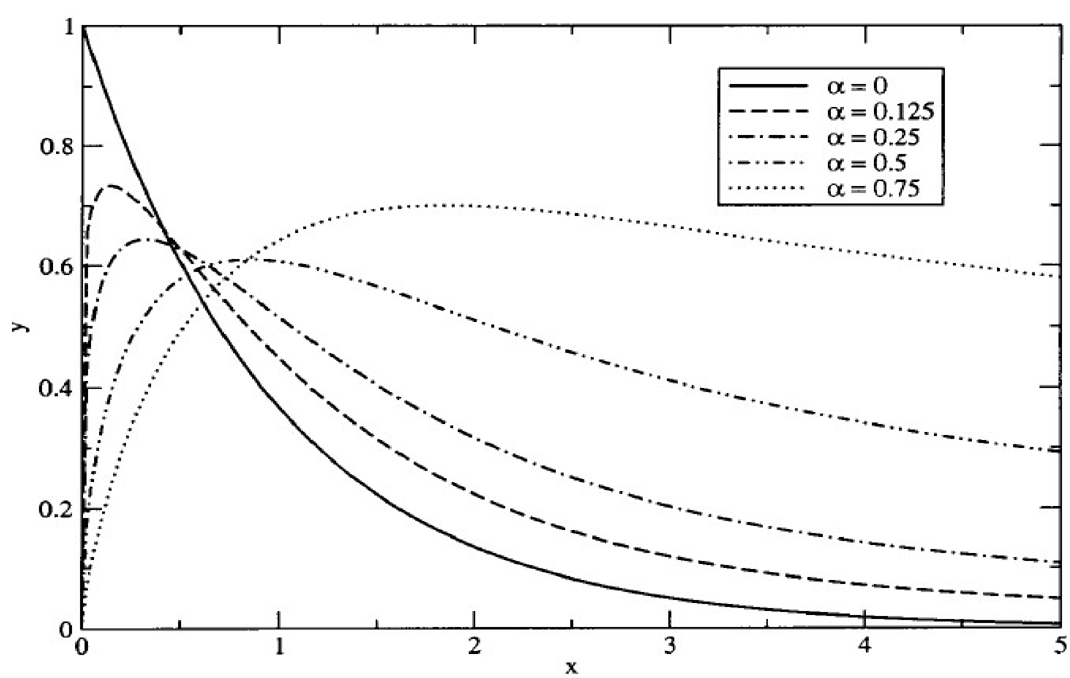

Figure 3.8.2: Riemann-Liouville integral I^{α} of y = e^{-x} for different values of α.

3.4 Caputo fractional derivative:

Italian mathematician M Caputo introduce Caputo fractional differential operator of order α > 0, m-1 < α < m for m ϵ N in 1967. The main advantage of using the Caputo definition is that it is easily interpreted initial conditions and it is also bounded, meaning that the derivative of a constant is equal to 0. The definition is as follows:

$$ {}^{C}D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{f^{m}(r)}{(x-r)^{\alpha+1-m}}dr, \quad (3.4.1)$$

Let α > 0 and m-1 < α < m, m ϵ N, and a < x < b, Left hand and Right hand Caputo Fractional derivative is defined as

$$ {}^{C}D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x (x-r)^{m-\alpha-1}f^{(m)}(r)dr, \quad (3.4.2)$$

$$ {}^{C}D_{b-}^{\alpha}f(x) = \frac{(-1)^{m}}{\Gamma(m-\alpha)}\int_x^b (x-r)^{m-\alpha-1}f^{(m)}(r)\,dr \quad (3.4.3)$$

Examples:

I) Evaluate ${}_{a}D_{x}^{\alpha}f(x)$, for α = ½ , a = 0 and f(x) = x

$$ {}_{0}D_{x}^{\frac{1}{2}}f(x) = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)}\int_0^x \frac{1}{(x-r)^{\frac{1}{2}}}dr$$

$$ {}_{0}D_{x}^{\frac{1}{2}}f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x \frac{1}{(x-r)^{\frac{1}{2}}}dr$$

$$ {}_{0}D_{x}^{\frac{1}{2}}f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^x (x-r)^{\frac{-1}{2}}dr$$

$$ {}_{0}D_{x}^{\frac{1}{2}}f(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)}\left[-2(x-r)^{\frac{1}{2}}\right]_0^x$$

$$ {}_{0}D_{x}^{\frac{1}{2}}f(x) = 2\sqrt{\frac{x}{\pi}}$$

II) Evaluate ${}_{a}D_{x}^{\alpha}f(x)$, for α = ½ , and f(x) = C, constant.

$$ {}_{a}D_{x}^{\frac{1}{2}}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{n}(r)}{(x-r)^{\alpha+1-n}}dr$$

$$ {}_{a}D_{x}^{\frac{1}{2}}C = \frac{1}{\Gamma\left(1-\frac{1}{2}\right)}\int_a^x \frac{0}{(x-r)^{\alpha+1-n}}dr$$

$$ {}_{a}D_{x}^{\frac{1}{2}}C = 0$$

Some properties of Caputo fractional Derivative:

I) Theorem 3.4.1: Let f(x) be a function and m-1 < α < m, α ϵ R such that its Caputo derivative ${}^{C}D^{\alpha}_{a+}f(x)$ exists, then prove

$${}^{C}_{a}D^{0}_{x}\, f(x) = f(x)$$

Proof: By Convention ${}^{C}_{a}D^{0}_{x}\, f(x) = f(x)$.

II) Theorem 3.4.2: Let f(x) & g(x) be any functions and m-1 < α < m, α ϵ R, λ ϵ C such that its Caputo derivative ${}^{C}D^{\alpha}_{a+}f(x)$ and ${}^{C}D^{\alpha}_{a+}g(x)$ exists, then prove

$${}^{C}_{a}D^{\alpha}_{x}\big(\lambda f(x) + g(x)\big) = \lambda\, {}^{C}_{a}D^{\alpha}_{x}f(x) + {}^{C}_{a}D^{\alpha}_{x}g(x)$$

Proof: ${}^{C}_{a}D^{\alpha}_{x}\big(\lambda f(x) + g(x)\big) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{\lambda f^{m}(r)+g^{m}(r)}{(x-r)^{\alpha+1-m}}dr$ by (3.4.1)

$$= \frac{1}{\Gamma(m-\alpha)}\left(\int_a^x \frac{\lambda f^{m}(r)}{(x-r)^{\alpha+1-m}}dr + \int_a^x \frac{g^{m}(r)}{(x-r)^{\alpha+1-m}}dr\right)$$

$$= \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{\lambda f^{m}(r)}{(x-r)^{\alpha+1-m}}dr + \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{g^{m}(r)}{(x-r)^{\alpha+1-m}}dr$$

$$= \frac{\lambda}{\Gamma(m-\alpha)}\int_a^x \frac{f^{m}(r)}{(x-r)^{\alpha+1-m}}dr + \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{g^{m}(r)}{(x-r)^{\alpha+1-m}}dr$$

$${}^{C}_{a}D^{\alpha}_{x}\big(\lambda f(x) + g(x)\big) = \lambda\, {}^{C}_{a}D^{\alpha}_{x}f(x) + {}^{C}_{a}D^{\alpha}_{x}g(x)$$

III) Theorem 3.4.3: Let f(x) be any functions and m-1 < α < m, α ϵ R, m, n ϵ N such that its Caputo derivative ${}^{C}_{a}D^{\alpha}_{x}f(x)$ exists, then prove

$$D^{n}\,{}^{C}_{a}D^{\alpha}_{x}f(x) = {}^{C}_{a}D^{n+\alpha}_{x}f(x)$$

Proof: use [3, 6]

Note: $D^{n}\,{}^{C}_{a}D^{\alpha}_{x}f(x) \neq {}^{C}_{a}D^{\alpha+n}_{x}f(x)$

Figure (3.4.1) shows graph of Caputo fractional derivative of function $I^{\alpha}f(x) = D^{-\alpha}$f(x) where f(x) = y = x for different values of α. Here we observe that for α = 0, 1 and all α > 1graph is in accordance with as an classical derivative.

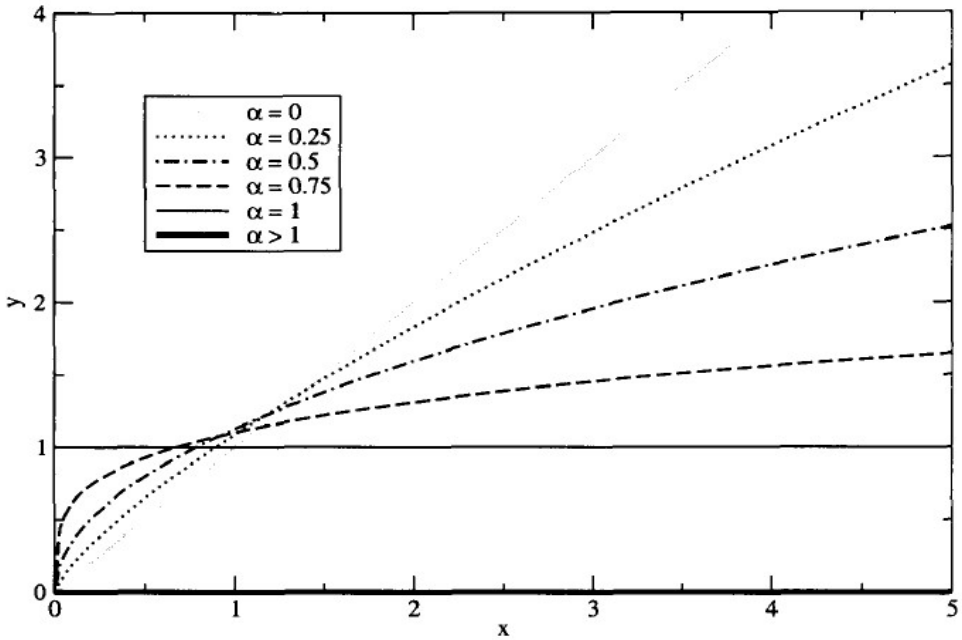

Figure 3.4.1: Caputo derivative ${}_aD_x^\alpha$ of y = x for different values of α.

Figure (3.4.2) show graph of Caputo fractional derivative of function $I^\alpha f(x) = D^{-\alpha}f(x)$ where f(x) = y = e^{-x} for different values of α between 0 and 1. These are mirror image of graphs in figure (2.8.2).

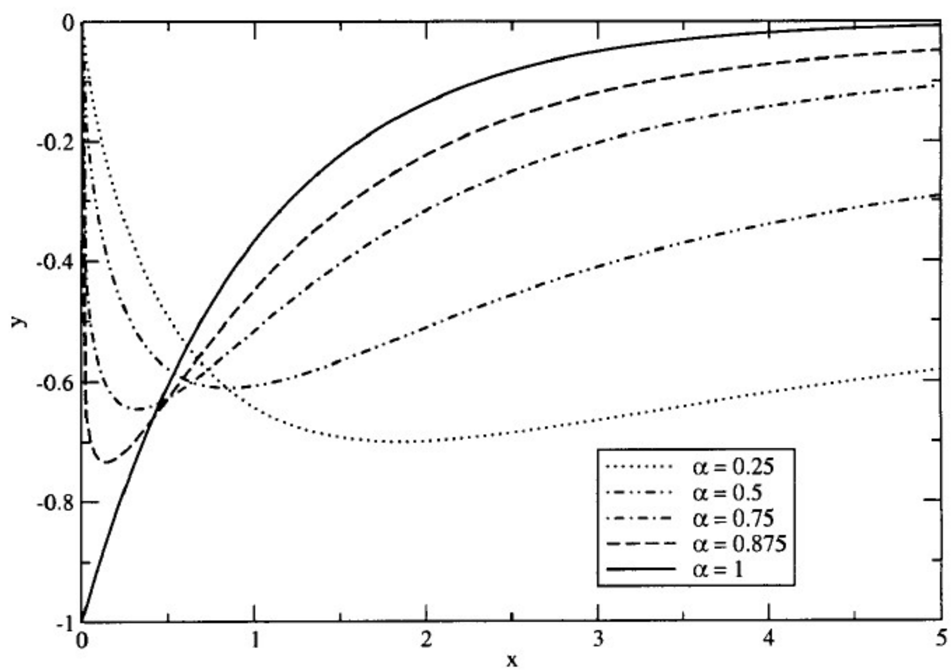

Figure 3.4.2: Caputo derivative ${}_aD_x^\alpha$ of y = e^{-x} for different values of α.

3.5 Comparative Study of Riemann-Liouville and Caputo fractional derivative:

Riemann-Liouville fractional derivative and Caputo fractional derivative are represented by equations (3.3.13) and (3.4.1) respectively. From these definitions following results holds.

I) Theorem 3.5.1: Let f(x) be a function and m-1 < α < m, α ϵ R such that its Caputo derivative ${}^{C}D_{a+}^{\alpha}f(x)$ exists, then prove

$$ {}^{C}D_{a+}^{\alpha}f(x) = I^{m-\alpha}D^{m}f(x) \quad (3.5.1)$$

Proof:

$$ {}^{C}D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{f^{m}(r)}{(x-r)^{\alpha+1-m}}dr$$

$$ {}^{C}D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(m-\alpha)}\int_a^x \frac{D^{m}f(r)}{(x-r)^{\alpha+1-m}}dr$$

$$ {}^{C}D_{a+}^{\alpha}f(x) = I^{m-\alpha}D^{m}f(x)$$

II) Theorem 3.5.2: Let f(x) be a function and m-1 < α < m, α ϵ R such that its Riemann-Liouville derivative $D_a^{\alpha}f(x)$ exists, then prove

$$D^{\alpha}f(x) = D^{m}\, I^{m-\alpha}f(x) \quad (3.5.2)$$

Proof:

$$ {}_{a}D_x^{\alpha}\,f(x) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^{m}\int_a^x (x-r)^{m-\alpha-1}f(r)\,dr$$

$$ {}_{a}D_x^{\alpha}\,f(x) = D^{m}\frac{1}{\Gamma(m-\alpha)}\int_a^x (x-r)^{m-\alpha-1}f(r)\,dr$$

$$ {}_{a}D_x^{\alpha}\,f(x) = D^{m}I^{m-\alpha}f(x) \quad \text{by (2.3.5)}$$

III) Theorem 3.5.3: Let f(x) be a function defined on (0, ∞) and m-1 < α < m, α ϵ R then following relation holds between Riemann-Liouville fractional derivative and Caputo fractional derivative

$$ {}^{C}D_{a+}^{\alpha}f(x) = D^{\alpha}f(x) - \sum_{k=o}^{m-1}\frac{r^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0) \quad (3.5.3)$$

Proof: By Taylor series expansion f(x) about point 0 can be written as

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{m-1}}{(m-1)!}f^{(m-1)}(0) + L_{m-1}$$

$$f(x) = \sum_{k=o}^{m-1}\frac{r^{k}}{\Gamma(k+1)}f^{(k)}(0) + L_{m-1} \quad (3.5.4)$$

$$L_{m-1} = \int_0^x \frac{f^{(m)}(r)(x-r)^{m-1}}{(m-1)!}dr$$

$$L_{m-1} = \frac{1}{\Gamma(m)}\int_0^x f^{(m)}(r)(x-r)^{m-1}\,dr$$

$$L_{m-1} = I^{m}f^{(m)}(x) \quad (3.5.5)$$

Applying RL derivative to equation (3.5.4) we have

$$D^{\alpha}f(x) = D^{\alpha}\left(\sum_{k=o}^{m-1}\frac{r^{k}}{\Gamma(k+1)}f^{(k)}(0)+L_{m-1}\right)$$

$$= \sum_{k=o}^{m-1}\frac{D^{\alpha}r^{k}}{\Gamma(k+1)}f^{(k)}(0)+D^{\alpha}L_{m-1}$$

$$= \sum_{k=o}^{m-1}\frac{\Gamma(k+1)r^{k-\alpha}}{\Gamma(k+1-\alpha)\Gamma(k+1)}f^{(k)}(0)+D^{\alpha}I^{m}f^{(m)}(x) \qquad \text{by (3.5.5)}$$

$$= \sum_{k=o}^{m-1}\frac{r^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0)+I^{m-\alpha}f^{(m)}(x)$$

$$= \sum_{k=o}^{m-1}\frac{r^{k-\alpha}}{\Gamma(k+1-\alpha)}f^{(k)}(0)+{}^{C}D_{a+}^{\alpha}f(x)$$

Hence, $^{C}D_{a+}^{\alpha}f(x) = D^{\alpha}f(x) - \sum_{k=o}^{m-1}\frac{r^{k-\alpha}}{\Gamma(k-\alpha+1)}f^{(k)}(0)$

Note: From equation (2.5.3), $^{C}D_{a+}^{\alpha}f(x) \neq D^{\alpha}f(x)$ i.e. Riemann-Liouville and Caputo derivatives do not coincide.

3.6 Other Approaches of fractional calculus:

FC has built-up several applications in countless areas of scientific knowledge. As a consequence, distinctive approaches to solve problems involving the derivative and integral were proposed. We do not talk about on the pros and cons of each approach and does not support that is to be differentiated or integrated. As like the above approaches which we discussed in previous part of this chapter which are Grünwald–Letnikov, Riemann-Liouville and Caputo for fractional integral and derivative, there are many approaches we can find in the literature. Some of these definitions for fractional integrals and fractional derivatives are

Fractional Integrals: Wely integral, Kober Integral, Hadamard Integral, Chen integral, Hilfer integral, Yang integral, Cossar integral, Erdelyi integral, Grünwald integral, Riemann integral, Letnikov integral, Liouville integrals, and many more.

Fractional Derivatives: Wely derivative, Hadamard derivative, Chen derivative, Marchaud derivative, Jumarie derivative, Cossar derivative, Yang derivative, Grünwald derivative, Riemann derivative, Letnikov derivative, Liouville derivative, and many more.

3.7 Conclusion:

In this chapter we studied mostly used fractional calculus operator right from their origin to their applications. From this study we can conclude that generalization of traditional calculus is an attractive topic for research, but due to its complexity, inability to satisfy all basic properties concerned with algebraic point of view, field is not developed as per requirement. There are numerous definitions available for fractional integral as well as for fractional derivative. Using these definitions researchers in this field developing it theoretically and via applicability in majority of fields in mathematics, science, engineering and other fields of education.

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CHAPTER -IV

EXTENDED TRANSFORMS AND FRACTIONAL DIFFERENTIAL EQUATIONS

4.1 Introduction:

Extended integral transform means an ordinary integral transform used to assess problems in fractional calculus. In evaluating problems in fractional calculus, fractional differential equations extended integral transform plays an imperative responsibility. Basic properties of these extended integral transform studied we initially overlook on ordinary integral transform.

4.2 Integral Transform:

An integral transform is a particular type of mathematical operator. To each integral transform there is its associated inverse integral transform. This inverse integral transform associated to respective integral transform since, integral transform maps its original domain into another domain where it is solved very easily than in its original domain than again mapped to original domain by using inverse integral transform. There are numerous integral transform in study.

The methods of Integral transforms have their genesis in nineteenth century work of Joseph Fourier and Oliver Heaviside. The fundamental idea is to represent a function f(x) in terms of a transform F(s) is

Integral Transform of the function f (x) for $t_1 \leq x \leq t_2$ is defined as

$$T(f(x)) = F(s) = \int_{t1}^{t2} f(t)K(t,x)dt$$

Where K is the function of two variables t and x called kernel of integral transform.

Choice of kernel function K of two variables is different for different integral transform.

Inverse integral transform is of the form

$$f(t) = \int_{x1}^{x2} T(f(x))\, K^{-1}(x,t)\, dx$$

Where, $K^{-1}(x,\ t)$ is the kernel of inverse integral transform and is the inverse of kernel K (t, x).

A) Laplace Transform:

Laplace Transform is very useful in solving differential equations especially when initial values are zero. The Laplace Transform of a function f(x) which is piecewise smooth over every finite interval [0, ∞) and of exponential order γ i.e. there exist constants M > 0 and X > 0 such that $|f(x)| \leq M\, e^{\gamma x}$ for all x > X, then the Laplace transform defined for all real number x ≥ 0 is the function $\mathcal{L}(f(x)) = F(s)$ given by

$$\mathcal{L}(f(x)) = F(s) = \int_0^{\infty} f(x)e^{-sx}\,dx \qquad (4.2.1)$$

Inverse Laplace transform of F(s) i.e. f (x) is given by

$$\mathcal{L}^{-1}(F(s)) = f(t) = \int_{c-i\infty}^{c+i\infty} e^{s}\, F(s)\, ds,\ c=Re(s) > c_0, \qquad (4.2.2)$$

where c_0 lies in right half plane of the absolute convergence of Laplace integral (4.2.1)

B) Some basic properties of Laplace Transform:

Theorem 4.3.1: Let f(x) and g(x) be two functions such that there Laplace transform F(s) and G(s) respectively exist, then following equations holds.

i) $\mathcal{L}\{a\, f(x)\}=a\, \mathcal{L}\{f(x)\}$; where a is constant.

ii) $\mathcal{L}\{a\, f(x) + b\, g(x)\}= a\, \mathcal{L}\{f(x)\} + b\, \mathcal{L}\{g(x)\}$; Linearity

iii) If $\mathcal{L}\{f(x)\}= F(s)$ then $\pounds\{f(ax)\} = \frac{1}{a} + F(\frac{s}{a})$; change of scale

iv) Convolution property : If F(s) and G(s) are the Laplace Transform of f(x) and g(x) respectively, then

$$\mathcal{L}\{ F(x) * g(x); s\} = F(s)\, G(s) = \mathcal{L}\left\{\int_0^x f(x-z)g(z)dz\right\},$$

Where convolution is given by

$$f * g = \left\{\int_0^x f(x-z)g(z)dz\right\}.$$

v) The Laplace transform of the n-th ($n \in N$) derivative of f(x) is given by

$$\mathcal{L}\{f^{(n)}(x):s\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0)$$
$$= s^n F(s) - \sum_{k=0}^{n-1} s^{k} f^{(n-k-1)}(0)$$

Proof: All proof are easily obtained from (4.2.1).

C) Mellin Transform:

Mellin Transform is closely related with Laplace transform and Fourier transform. The Mellin integral transform was employed in connection with the Fractional Calculus special functions, like the Mittag-Leffler and the Wright functions and their generalizations. These functions are scrupulous cases of the Fox *H*-function that can be interpreted as an inverse Mellin transform. Hjalmar Mellin (1854–1933) gave his name to the Mellin transform of a function *f*(*x*) defined over the positive real's, the complex function M [f (x); s]. Mellin Transform is the multiplicative version of two sided Laplace Transform.

The mellin Transform of a function f(x) is defined as

$$\mathcal{M}\{f(x)\} = F(s) = \int_0^\infty x^{s-1} f(x)\, dx \qquad (4.2.3)$$

Where $\mathcal{M}$ is the the Mellin transform operator and s is the Mellin transform variable which is complex number.

Inverse mellin transform is defined as

$$\mathcal{M}^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s)\, ds \qquad (4.2.4)$$

Where $\mathcal{M}^{-1}$ is the inverse Mellin transform operator, and integral is understood in the good judgment of Cauchy principal value.

D) Some properties of Mellin Transform:

Theorem 4.2.2: Let f(x) and g(x) be two functions such that there Millin transform F(s) and G(s) respectively exist, then following equations holds

i) .$\mathcal{M}\{a\, f(x)\} = a\, \mathcal{M}\{f(x)\}$; where a is constant

ii) $\mathcal{M}\{a\, f(x) + b\, g(x)\} = a\, \mathcal{M}\{f(x)\} + b\, \mathcal{M}\{g(x)\}$; Linearity

iii) If $\mathcal{M}\{f(x)\} = F(s)$ then $\mathcal{M}\{f(ax)\} = a^{-s}\, F(s)$; change of scale

iv) Convolution property : If F(s) and G(s) are the Mellin Transform of f(x) and g(x) respectively, then

$$\mathcal{M}\{ f(x) * g(x); s\} = F(s)\, G(s) = \left\{\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(s-z) g(z) dz\right\}$$

Where convolution is given by

$$f * g = \left\{\int_0^x f(x-z) g(z) dz\right\}.$$

v) The Mellin transform of the n-th ($n \in N$) derivative of f(x) is given by

$$\mathcal{M}\left\{f^{(n)}(x) : s\right\} = \int_0^\infty f^n(x) x^{s-1} dx = (-1)^k \frac{\Gamma(s)}{\Gamma(s-k)} F(s-k)$$

vi) The Mellin transform of product of x^n and the n-th ($n \in N$) derivative of f(x) is given by

$$\mathcal{M}\left\{x^n f^{(n)}(x) : s\right\} = \int_0^\infty x^n f^n(x) x^{s-1} dx = (-s)^n \mathcal{M}\, [f(x) : s]$$

vii) $$\left(\frac{d}{ds}\right)^n \mathcal{M}\, [f(x) : s] = \int_0^\infty f(x) (\log x)^n x^{s-1} dx$$

$$= \mathcal{M}\, [(\log x)^n f(x) : s-1]$$

Proof: All proof are easily obtained from (4.2.3) see [12, 13].

4.3 Extended Transform (Integral transform of some fractional approaches):

I) Laplace Transform of the Riemann-Liouville Fractional Integral:

Theorem 4.3.1: Suppose α > 0 and F(s) is the Laplace transform of f(x), then Laplace transform of the Riemann-Liouville Fractional Integral

$$D^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt. \quad (4.3.1)$$

is given by

$$\mathcal{L}\{D^{-\alpha}f(x)\} = \frac{1}{\Gamma(\alpha)}\mathcal{L}\{x^{\alpha-1}\}\mathcal{L}\{f(x)\} = s^{-\alpha}F(s) \quad (4.3.2)$$

Proof: To obtain Laplace transform of (4.3.1)

Following table gives Laplace transform of some function and their fractional integrals:

f(x)	F(s)= £{f(x)}	$D^{-\alpha}\{f(x)\}$	$£\{D^{-\alpha}f(x)\}$
e^{ax}	$\frac{1}{s-a}$	$D^{-\alpha}\{e^{ax}\}$	$\frac{1}{s^{\alpha}(s-a)}$
x^{v}	$\frac{\Gamma(v+1)}{s^{v+1}}$	$D^{-\alpha}\{x^{v}\}$	$\frac{\Gamma(v+1)}{s^{\alpha+v+1}}$
$x^{v-1}e^{ax}$	$\frac{\Gamma(v)}{(s-a)^{v}}$	$D^{-\alpha}\{x^{v-1}e^{ax}\}$	$\frac{\Gamma(v)}{s^{\alpha}(s-a)^{v}}$
Cos(ax)	$\frac{s}{s^2+a^2}$	$D^{-\alpha}\{\cos(ax)\}$	$\frac{s}{s^{\alpha}(s^2+a^2)}$
Sin(ax)	$\frac{a}{s^2+a^2}$	$D^{-\alpha}\{\sin(ax)\}$	$\frac{a}{s^{\alpha}(s^2+a^2)}$

Table 4.2.1: Fractional Laplace transform of some functions.

II) Laplace Transform of the Riemann-Liouville Fractional Derivative:

Theorem 4.3.1: Suppose α > 0 and F(s) is the Laplace transform of f(x), then Laplace transform of the Riemann-Liouville Fractional differential operator

$$D^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_a^x (x-t)^{n-\alpha-1} f(t)\, dt \quad (4.3.3)$$

Is given by $\mathcal{L}\{D^{\alpha}f(x):s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{k}\left[D^{(\alpha-k-1)}f(0)\right]$

$$= s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{n-k-1}[D^k I^{n-\alpha} f(0)] \qquad (4.3.4)$$

Proof: To obtain Laplace transform of (4.3.3) see [11, 12].

III) Laplace Transform of the Caputo Fractional Derivative:

Theorem 4.3.1: Suppose α > 0 and F(s) is the Laplace transform of f(x), then Laplace transform of the The Caputo Fractional Differential operator is given by

$${}_a^cD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)\, dt, \qquad (4.3.5)$$

is given by

$$\mathcal{L}\{{}_a^cD_x^{\alpha} f(x) : s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n. \qquad (4.3.6)$$

Proof: To obtain Laplace transform of (4.3.5) we have

$${}_a^cD_x^{\alpha} f(x) = I^{n-\alpha} D^n f(x)$$

Let $g(x) = D^n f(x)$

$${}_a^cD_x^{\alpha} f(x) = I^{n-\alpha} g(x)$$

$$\mathcal{L}\{{}_a^cD_x^{\alpha} f(x)\} = \mathcal{L}\{I^{n-\alpha} g(x)\}$$

$$= s^{-(n-\alpha)} G(x) \qquad (4.3.7)$$

Where $G(x) = \mathcal{L}\{g(x)\}$ given by from (4.3.4)

$$G(x) = s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}[D^k f(0)] \qquad (4.3.8)$$

Using (4.3.8), equation (4.3.7) becomes

$$\mathcal{L}\{{}_a^cD_x^{\alpha} f(x)\} = s^{-(n-\alpha)}\left(s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1}[D^k f(0)]\right)$$

Hence, $\mathcal{L}\{{}_a^cD_x^{\alpha} f(x) : s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$

IV) Mellin Transform of the Riemann-Liouville Fractional Integral:

Mellin transform of Riemann-Liouville fractional integral operator is given by

$$\mathcal{M}\{I^{\alpha}\mathbf{f}(x)\} = \mathcal{M}\{D^{-\alpha}\mathbf{f}(x)\} = \mathrm{F}(s) = \frac{\Gamma(1-s-\alpha)}{\Gamma(1-s)} \mathrm{F}(s+\alpha) \qquad (5.5.4)$$

V) Mellin Transform of the Riemann-Liouville and Caputo Fractional Derivative:

Mellin transform of Riemann-Liouville and Caputo fractional derivative operator is given by

$$\mathcal{M}\{D^{\alpha}\mathrm{f}(x)\} = \mathrm{F}(s) = \frac{\Gamma(1-s+\alpha)}{\Gamma(1-s)} \mathrm{F}(s-\alpha) \qquad (5.5.5)$$

Theorem 5.5.2: Let f be Mellin transformable function for all x ≥ 0, and 0 ≤ n-1 ≤ α ≤ n then following properties holds.

i) $\mathcal{M}[D^{\alpha}I^{\alpha}f(x);s] = \mathcal{M}[f(x);s]$

ii) $\mathcal{M}[I^{\alpha}D^{\alpha}f(x);s] = \mathcal{M}[f(x);s] - \sum_{k=0}^{n-1}\frac{f^{k}(0)}{k!(k+s)}$, Re(s) > - Re(k)

iii) $\mathcal{M}[I^{\alpha}I^{\beta}f(x);s] = \frac{\Gamma(1-\alpha-\beta-s)}{\Gamma(1-s)}\mathcal{M}[f(x);\ \alpha+\beta+s]$

Proof: i) Here $D^{\alpha}$is the Caputo fractional derivative of order α and I^{α} is the Riemann-Liouville fractional integral. We know a result

$D^{\alpha}[I^{\alpha}f(x)] = f(x)$

Using this result

$\mathcal{M}[D^{\alpha}I^{\alpha}f(x);s] = \mathcal{M}[f(x);s]$

ii) Here $D^{\alpha}$is the Caputo fractional derivative of order α and I^{α} is the Riemann-Liouville fractional integral. We know a result

$$I^{\alpha}[\ D^{\alpha}f(x)] = f(x) - \sum_{k=0}^{n-1}\frac{f^{k}(0)}{k!}x^{k}$$

Using this result, we have

$$\mathcal{M}[I^{\alpha}D^{\alpha}f(x);s] = \mathcal{M}[f(x);s] - \mathcal{M}\left[\sum_{k=0}^{n-1}\frac{f^{k}(0)}{k!}x^{k};s\right]$$

$$= \mathcal{M}[f(x);s] - \sum_{k=0}^{n-1}\frac{f^{k}(0)}{k!}\int_{0}^{\infty} x^{k+s-1}\,dx$$

$$= \mathcal{M}[f(x);s] - \sum_{k=0}^{n-1}\frac{f^{k}(0)}{k!(k+s)},\quad \text{Re(s) > - Re(k)}$$

iii) Consider

$$\mathcal{M}[I^{\alpha}I^{\beta}f(x);s] = \mathcal{M}[I^{\alpha+\beta}f(x);s]$$

$$= \frac{\Gamma(1-\alpha-\beta-s)}{\Gamma(1-s)}\mathcal{M}[f(x);\ \alpha+\beta+s] \quad \text{by (5.5.4)}$$

4.4 Fractional Differential Equations:

The laws of the Natural and Physical world are usually modeled in mathematics in the form of differential equations. Fractional Calculus is the generalization of traditional calculus therefore fractional differential equation gives more energy for modeling real world problems than that of differential equations. There is no standard algorithm to solve fractional differential equations. Solution of the fractional differential equations and its interpretation is a rising field of Applied Mathematics. Most of the fractional differential equations do not have exact analytic solutions therefore numerical techniques and approximation methods are used. Here we study some of these methods/ techniques and tried to analyze them.

Ordinary differential calculus and Fractional differential calculus has lot of differences between them such as Ordinary differential calculus (ODC) initiated at first then fractional differential calculus (FDC), FDC is the generalization of ODC but not vice-versa, development of ODC was very fast where as development of FDC was very slow, ODC easy to interpret (most of the time) geometrically and physically where as FDC very hard to interpret geometrically as well as physically, ODC termed as local and FDC termed as non-local, ODC models ideal behavior where as FDC models real behavior etc.

In last two to three decades we find numerous applications of Fractional differential equations (FDE) in all most all fields of science and engineering such as Visco-elasticity, Control theory, Biology, physics, chemical sciences, engineering, electric circuits, bioengineering etc. Fractional differential equations are used to model most of physical phenomenon hence fraction differential equation are the solutions to real world problems. Several methods are used to solve fractional differential equations. Some of them are the Adomian decomposition method, variational iteration method, Homotopy Analysis method, integral transform method, the iteration method, the operational method and new iterative method (Daftardar-Gejji and Jafri (2006)) etc.

Fractional differential equations have gained considerable significance due to their frequent form applications in fluid flow, dynamical processes in self similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics, control theory of dynamical systems, optics, viscoelasticity, electrochemistry of corrosion, chemical physics, and signal processing, and so on. These applications in interdisciplinary sciences motivate us to try to find out the analytic or numerical solutions for the fractional differential equations. But for most ones it is difficult to find out or even have exact solutions. Thus, necessarily, the numerical techniques are applied to the fractional differential equations. Now, many effective methods for solving fractional differential equations have been presented, such as nonlinear functional analysis method including monotone iterative technique, topological degree theory, and fixed point theorems. Also, numerical solutions are obtained by the following methods: random walk, matrix approach, the Adomian decomposition method and variational iteration method, HAM.

In opposite to differential equations of integer order, in which derivatives depend only on the local behaviour of the function, fractional differential equations accumulate the whole information of the function in a weighted form. This is so called memory effect and has many applications in physics, chemistry, engineering, etc. For that reason we need a method for solving such equations which will be effective, easy-to-use and applied for the equations in general form. However, known methods used for solution of the equations have more disadvantages. Analytical methods, described in detail in, do not work in the case of arbitrary real order. Another analytical method, which uses the multivariate Mittag-Leffler function and generalizes the previous results, can be used only for linear type of equations. On the other hand, for specific differential equations with oscillating and periodic solution there are some specific numerical methods. Other numerical methods allow solution of the equations of arbitrary real order but they work properly only for relatively simple form of fractional equations.

4.5 Some Approaches for solving FDE:

Consider non-linear functional equation

$$u = f + L(u) + N(u) \qquad (4.5.1)$$

where L and N are linear and nonlinear functions of u and f is a source function. This equation represents ordinary differential equations (ODE), integral equations, partial differential equations (PDE), differential equations involving fractional order, and so on. Various methods such as Green's function method, Laplace transform method and Fourier transform method, have been used to solve linear equations. For solving non-linear equations however, one has the option to decomposition, numerical or iterative methods.

Among the several numbers of available approaches for solving fractional differential equation here we review following three mostly used approaches.

I) Adomian Decomposition Method:

The Adomian decomposition method is a powerful technique used to solve nonlinear functional equation in particular fractional differential equations. This method provides efficient algorithms for numeric simulation and analytic approximation of solution to considered fractional differential equations. For finding analytic solution to nonlinear functional equations Adomian decomposition method is an accurate, convenient and effective. In Adomian decomposition method given problem/equation split into linear and nonlinear part, then inverting highest order derivative operator contained in the linear operator in both sides. For detail understanding consider a general nonlinear equation in the form

$$Lu + Ru + Nu = g$$

Where L is the highest order derivative which is to be invertible easily, R is the linear differential operator having order less then L, Nu represent the nonlinear terms and g is the source term. Applying L^{-1} to both sides, we have

$$u = f(x) - L^{-1}(Ru) - L^{-1}(Nu)$$

where $f(x) = L^{-1}g(x)$ with given conditions.

For nonlinear differential equations Nu = F(u) and

$F(u) = \sum_{n=0}^{\infty} A_n$, infinite series of Adomian Polynomials.

Where A_n is called Adomian polynomial of $u_0, u_1, \ldots\ldots u_n$ defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{i=0}^{n}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n = 0, 1, 2, \ldots..$$

$u = \sum_{n=0}^{\infty} u_n$, is the approximate solution.

Adomian decomposition method (ADM) has proved to be a useful tool for solving functional equations, since it offers certain advantages over numerical methods. Adomian's technique is simple in principle but involves tedious calculations of Adomian polynomials.

II) Variational Iteration Method:

Variational iteration method is a nice method used to approximate analytical solutions of both linear and nonlinear fractional differential equations. Variational iteration method first successfully applied by Ji-Huan He to the fractional differential equations. Large number of fractional differential equations does not have exact analytic solutions, so approximation by variational iteration method is an best option/deal. Most authors found that the shortcomings arising in the Adomian decomposing method can be completely eliminated by the variational iteration method. For example, Abbasbandy applied the variational iteration method to Riemann–Liouville's fractional derivatives, Draganescu and his colleagues to nonlinear vibration with fractional damping, Momani and his colleagues applied the method to fluid mechanics where the fractional derivative was successfully applied.

General form of variational iteration method

$$Lu(x) + N(x) = g(x)$$

Where L is the linear operator and N is the nonlinear operator and g(x) is a known analytic function.

III) Homotopy Analysis Method:

S.J. Liao introduced Homotopy Analysis Method to obtain solution to linear/non-linear differential equations. Homotopy analysis method is a general approximate analytic approach used to obtain series solution of nonlinear equations of different type. This method is applied to various nonlinear problems in science and engineering by number of researchers.

In homotopy analysis approach we construct a continuous mapping of an initial guessed approximation to the exact solution of the equation which is in consideration. Continuous mapping constructed by choosing proper auxiliary linear operator and auxiliary parameter is used to ensure the convergence of the solution series.

4.6 Analysis of Methods:

Above three methods applied to fractional differential equation of the type given by

$D_*^{\gamma_i} x_i(t) = f_i\ (t, x_1, x_2, \dots, x_n, x_1', x_2', \dots\dots, x_n'),$

$i = 1, 2, \dots\dots, n-1,$

$0 < \gamma_i \le 1, t \ge 0$

$$g(t, x_1, x_2, \dots, x_n) = 0$$

Subject to the initial conditions

$$x_i(0) = a_i,\ i = 1, 2, \dots\dots, n.$$

Where $D_*^{\gamma_i}$ is Caputo differential operator as given in equation (4).

This fractional differential equation is converted into the required format of corresponding method and then evaluated to get solution.

A numerical example solved by these methods and data is collected in table4.5.1, table4.5.2, table 4.5.3 for different value of γ_i

$$D_*^{\gamma} x(t) - t y'(t) + x(t) - (1 + t) y(t) = 0, \quad 0 < \gamma \le 1,$$

$y(t) - \sin(t) = 0$

With initial condition x(0)=1, y(0)=0.

This differential has exact solution

$x(t) = e^{-t} + t sint\ , y(t) = \sin(t)$ when $\gamma = 1$.

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t	$\gamma_i = 0.5$		
	u_{HAM}	u_{ADM}	u_{VIM}
0	1.00000000	1.00000000	1.00000000
0.2	0.75450966	0.75450964	0.75450964
0.4	0.85249503	0.85249505	0.85249504
0.6	1.02420516	1.02420516	1.02420516
0.8	1.22732911	1.22732911	1.22732910
1.0	1.43275523	1.43275529	1.43275528

Table 4.5.1: Numerical solution to example for $\gamma_i = 0.5$

t	$\gamma_i = 0.75$		
	u_{HAM}	u_{ADM}	u_{VIM}
0	1.00000000	1.00000000	1.00000000
0.2	0.80166961	0.80166972	0.80166972
0.4	0.82508732	0.82508712	0.82508712
0.6	0.94545818	0.94545815	0.94545815
0.8	1.12295920	1.12295946	1.12295946
1.0	1.32697566	1.32697590	1.32697590

Table 4.5.2: Numerical solution to example for $\gamma_i = 0.75$

t	$\gamma_i = 1$			
	u_{HAM}	u_{ADM}	u_{VIM}	u_{exact}
0	1.00000000	1.00000000	1.00000000	1.00000000
0.2	0.85846462	0.85846462	0.85846462	0.85846462
0.4	0.82608738	0.82608738	0.82608738	0.82608739
0.6	0.88759712	0.88759712	0.88759712	0.88759712
0.8	1.02321384	1.02321384	1.02321384	1.02321384
1.0	1.20935045	1.20935045	1.20935045	1.20935043

Table 4.5.3: Numerical solution to example for $\gamma_i = 1$

4.6 Conclusion:

Applications of Fractional calculus are found in almost all sciences. Fractional Calculus is the topic of today's researchers. Fractional calculus is expressed as a solution to real world problems. Integral transform are useful in finding solution of differential equations. Solving fractional differential equation is very monotonous mission. We studied extended integral transform with their properties particularly Laplace transform and Mellin transform of Riemann-Liouville integral operator, Riemann-Liouville differential operator and Caputo differential operator.

All of these three methods, Adomian decomposition method, Variational iteration method and Homotopy analysis method gives approximately same solution. Adomian decomposition method and variational iteration method are simple and very easy to apply then homotopy iteration method.

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Chapter -V
Applications of Fractional Calculus

5.1 Introduction:

In recent year Mathematical applications using fractional calculus are increases speedily, in this section we see a quantity of real life mathematical modeling using fractional calculus. Fractional Calculus has engrossed concentration of many researchers, since fractional Calculus models are extra realistic and matter-of-fact than the classical integer order models. Therefore, any dynamical process modeled through fractional order differential equations has a reminiscence or memory effect.

5.2 Mathematical modeling using fractional calculus:

$P\, I^{\lambda}\, D^{\mu}$ Controller:

PID controller is also called as three term controller which is widely used in industrial control system. PID controller incessantly calculates an error value e(x) as the difference between a preferred setpoint (sp) and a calculated process variable (pv). It was used for automatic process control in the manufacturing industry. In today's era concept of PID is used universally where there is requirement of accurate and optimized automatic control.

The concept of a fractional order $\mathrm{P\, I^{\lambda}\, D^{\mu}}$ is proposed in a paper written by Igor Podlubny in 1999 [1], where the integrator and differentiator are of a fractional order. A fractional order transfer function is provided as

$$(s) = \frac{U(s)}{E(s)} = K_p + K_I s^{-\lambda} + K_D s^{\mu} \qquad \lambda, \mu > 0$$

Here λ is the order of the integrator, μ is the order of differentiator, $G_c(s)$ is the transfer of controller, U(s) is the controller's output and E(s) is an error. If λ =1 and μ = 1 equation becomes traditional P I D controller equation If λ =1 and μ = 0, equation converts to a P I controller equation. If λ =0 and μ = 1 equation converts to a P D controller equation.

In the time domain, it becomes an open-loop system is described by

$$\sum_{k=0}^{n} a_k \mathcal{D}^{\beta_k} y(t) = K_p w(t) + K_I \mathcal{D}^{-\lambda} w(t) + K_D \mathcal{D}^{\mu} w(t)$$

Here w(t) is the input , y(t) is output of the system, β_k (k= 0,1,2….n) arbitrary real number and a_k (k= 0,1,2….n) arbitrary constants. Effectiveness of this controller can be analyzes by an example of PD^{μ} controller. The transfer functions and time domain fractional order differential equation are

$$G(s) = \frac{1}{a_2 s^{\beta_2}} + a_1 s^{\beta_1} + a_0$$

$$a_2 y^{\beta_2}(t) + a_1 y^{\beta_1}(t) + a_0 y(t) = u(t)$$

With initial condition y(0) = 0, y'(0) = 0, y''(0) = 0.

Following figure shows the effectiveness of the controllers.

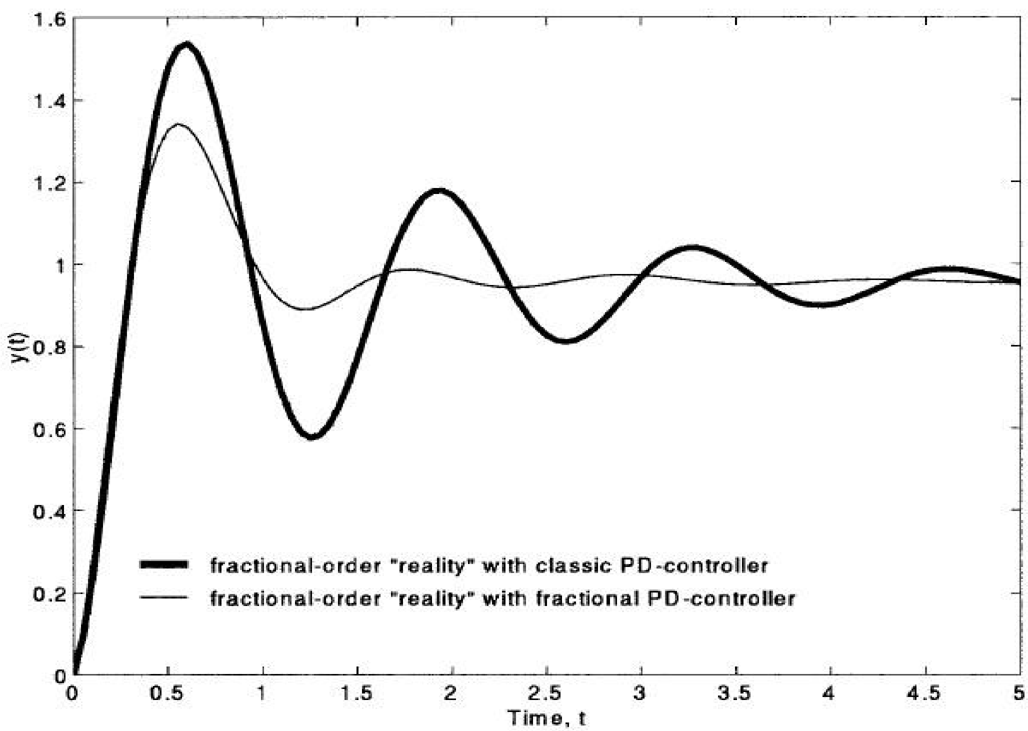

Figure 1 is the comparison of conventional P D controller (thick line) and fractional PD^{μ} controller (thin line).

Memory for propagation of computer viruses under human intervention:

Internet is now a necessary part of human life, without internet day today's life crumple. Use of internet means use of Computers, Laptops, mobiles etc. Keeping computers, laptops, Mobile etc in good/working condition is an important task. Most of the work such as banking, paying bills, purchasing some product online, online recharging of mobile, TV, booking movie ticket, etc requires internet. While downloading some file or transferring file from infected USB to computers, viruses may insert in the system and they may create problems to system. Computer viruses are malicious codes that can replicate themselves and spread among computers in network. These viruses create problems in proper working of computer system which result in disturbing routine working. Large number of companies, organizations etc are suffered, suffering and may be suffered from such viruses. 'My Doom' is the most devastating computer virus which caused over $38 billion on damages.

Human intervention plays a significant role in preventing the breakout of computer viruse. Here we study fractional mathematical model with memory propagation of computer viruses under human intervention. In this fractional model Caputo fractional derivative, Riemann-Liouville fractional derivative and Grunwald-Letnikov fractional derivatives are proposed.

This model is based on integer order model. Computers under consideration are categorized into three populations: Infected computers I(t) ; susceptible (virus free) computers S(t) and Recovered computers R(t) which are virus free computers but having some immunity. These variables used to develop model and virus-free equilibrium point and its stability, existence of uniformly stable solution and by using predicator corrector method, numerical results obtained.

Smoking dynamics using fractional differential equations:

We are all well knows that lot of health problems occurs due to tobacco smoking. Some harmful diseases due to smoking are cancer, stomach ulcer, high blood pressure, lung disease, heart disease etc. To reduce or to keep control on the strength of smokers all over the world, different mathematical models are proposed and also working-on to propose by some mathematicians. First simple mathematical model was proposed by C. Castillo-Garsow et.al. for giving up smoking. On considering control variables in the form of anti-smoking gum, anti-nicotine medicine/drugs, education campaign, eradication of smoking in a community, optimal control theory was proposed in. A novel model was proposed by assuming variables for mild smokers and chain smoker classes by Sharmi and Gumel. Smoking behavior under influence of education program and individuals determination to quit smoking was proposed.

Considering the Caputo- Fabrizio-Caputo fractional derivative, the authors in presented a new fractional giving up smoking model, the existence and uniqueness of the solution were discussed by the fixed point postulate. Zeb in proposed a fractional smoking dynamic model considering adolescent nicotine dependence. In the authors studied the giving up smoking dynamics using a fractional order model; approximate solutions via Laplace Adomian decomposition method were obtained. The multi-step generalized differential transform method was employed in to obtain accurate solutions to a giving up smoking model of fractional order. The giving up smoking dynamics models have been extended to the scope of fractional derivatives using power law and exponential decay law.

Here we study a mathematical model using fractional calculus i.e. fractional differential equation with local and non-local kernel for smoking dynamics which was proposed by V.F.Morals-Delgado et.al. In this model analytical solution obtained using Modified Homotopy Analysis Transform Method (MHATM) with two wisdom, one using Liouville – Caputo fractional derivative and second Atangana-Baleanu-Caputo fractional derivative. Also using iterative method through Laplace transform, special solution were obtained.

Liouville Caputo Fractional derivative is given by

$$ {}_{t_0}^{c}D_t^{\alpha}\{f(x)\} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} f'(t)\,(x-t)^{-\alpha} dt \,, $$

Where $\Gamma(.)$ denote the Euler's gamma function.

Laplace transform of Liouville-Caputo derivative is given by

$$ \mathcal{L}\{{}_{t_n}^{c}D_t^{\alpha}\{f(x)\}: s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha < n. $$

Atangana-Baleanu-Caputo (ABC) [18] fractional derivative is given by

$$ {}_{t_0}^{ABC}D_t^{\alpha}\{f(x)\} = \frac{B(\alpha)}{1-\alpha} \int_{t_n}^{t} f'(t)\, E_{\alpha}\left[-\alpha\frac{(x-t)^{\alpha}}{1-\alpha}\right] dt \,, \quad n-1 < \alpha(t) \leq n. $$

Where B(α) denote a normalize function and $E_{\alpha}(.)$ denote Mittag-Leffler function.

Laplace transform of Atangana-Baleanu-Caputo (ABC) fractional derivative is given by

$$ \mathcal{L}\{{}_{t_0}^{ABC}D_t^{\alpha}\{f(x)\}: s\} = \frac{B(\alpha)}{1-\alpha}\, \mathcal{L}\left[\int_{t_n}^{t} f'(t)\, E_{\alpha}\left[-\alpha\frac{(x-t)^{\alpha}}{1-\alpha}\right] dt\right](s). $$

Mathematical model developed by author's contains power law and fractional differentiation involving generalized Mittag Leffler functionas karnel due to non-locality of the model. Fixed point theorem and Picard-Lindelof approach used to put remarks on existence and uniqueness of system of solutions.

Diffusion Equation:

Diffusion equation is an interesting application of fractional calculus. The study of thermal flux on a given surface is important due to its influence on material wear and performance. In addition once the thermal flux is known, the temperature can be obtained. The brake disks are treated as semi-infinite bodies and assumed to have a constant temperature distribution.

Agrawal (2004) published a paper which analyzes the effectiveness of using fractional order derivatives to obtain the heat flux at a given point. Traditionally this was achieved by performing a transient analysis of two nearby points. His motive was the thermal study of disk brakes. The following diffusion equations govern the thermal distribution of the body.

$$\frac{\partial T(x,t)}{\partial t} = \frac{K}{\rho c}\frac{\partial^2 T(x,t)}{\partial x^2}$$

Where T(x, t) is the temperature at point x and time t, K is the thermal conductivity, ρ the mass density and c the specific heat of the disk material. After non-dimensional zing and applying Laplace Transform it is converted in fractional partial differential equation given by

$$\frac{1}{\sqrt{\alpha}}\frac{\partial^{1/2}\theta(x,t)}{\partial t^{1/2}} = \frac{\partial\theta(x,t)}{\partial x}$$

Using this fractional equation heat flux Q(t) and temperature at that point obtained.

Lot of Mathematicians work on diffusion equation some of them, Kulish gives more information on thermal flux analysis with fractional order derivatives in his paper, Lokenath Debnath also gives more detailed applications of fractional calculus relating to the diffusion equation in his litreture.

Resistance, Inductance and Capacitance Circuit:

RLC electrical circuit with a capacitor and an inductor are connected in parallel and this set is connected in series with a resistor and voltage. The capacitance *C*, the inductance *L* and the resistor *R* are consider positive constants and ϕ(x) is the ramp function, consider the ϕ(x) is Heaviside function.

The equations connected with a three elements of *RLC* electrical circuit are

The Voltage drop across resistor

$VD_R(x) = RI(x)$

The Voltage drop across inductor

$VD_L(x) = L\frac{d}{dx}I(x)$

The Voltage drop across capacitor

$VD_c(x) = L\frac{d}{dx}I(x)$

Where I(x) is the current in circuit.

Applying the Kirchhoff's voltage law and the equations associated with the three elements, we can write the non-homogeneous second order ordinary differential equation

$$L\frac{d^2}{dx^2}VD_c(x) + R\frac{d}{dx}VD_c(x) + \frac{1}{L}VD_c(x) = \frac{d}{dx}\phi(x) \qquad (1)$$

Similarly we obtain other non-homogeneous second order ordinary differential equations associated with the current on the capaciter,

$$\frac{L}{C}\frac{d}{dx}I_c(x) + \frac{R}{C}I_c(x) + \frac{1}{C}\int_0^x I_c(t)dt = \frac{d}{dx}\phi(x) \qquad (2)$$

We consider the initial condition $I_c(0) = 0$ and the solution can be establish in provisions of an exponential function. Fractional integro-differential equations of (2) is given by

$$\frac{L}{C}\frac{d^\alpha}{dx^\alpha}I_c(x) + \frac{R}{C}I_c(x) + \frac{1}{C}\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}I_c(t)dt = \frac{d}{dx}\phi(x) \qquad (3)$$

With $0 < \alpha \leq 1$, and the fractional derivative is used in the Caputo form, where $\phi(x)$ is the Heaviside function. We also consider $I_c(0) = 0$, i.e., the initial current on the capacitor is zero. We note that this equation is a achievable generalization of the classical integro-differential equation associated with the *RLC* electrical circuit, when $\alpha = 1$.

To solve this fractional integro-differential equation, we introduce the Laplace integral transform, defined by

$$L[I_c(x)] = F(s) = \int_0^\infty e^{-st} I_c(t)dt \qquad (4)$$

Equation (2.2.3) for R(s) > 0 becomes,

$$\frac{L}{C}s^\alpha F(s) + \frac{R}{C}F(s) + \frac{1}{C}\frac{F(s)}{s^\alpha} = 1 \qquad (5)$$

$$F(s) = \frac{C}{L}\frac{s^\alpha}{s^{2\alpha} + as^\alpha + b} \qquad (6)$$

Where a= R/L and b=1/L with a, b > 0

To get solution to (3) taking Laplace inverse of (5)

$$I_c(x) = \frac{C}{L} L^{-1}\left[\frac{s^{\alpha}}{s^{2\alpha}+as^{\alpha}+b}\right] \qquad (7)$$

$$L^{-1}\left[\frac{s^{\gamma-1}}{s^{\alpha}+As^{\beta}+B}\right] = x^{\alpha-\gamma}\sum_{r=0}^{\infty}(-A)^r x^{(\alpha-\beta)r} E_{\alpha,\alpha+1-\gamma+(\alpha-\beta)r}^{r+1}(-Bx^{r})$$

Valid when $\left|\frac{As^{\beta}}{s^{\alpha}+B}\right| < 1 \; and \; \alpha \geq \beta$, hence equation (2.2.7) will be

$$I_c(x) = \frac{C}{L} x^{\alpha-1} \sum_{r=0}^{\infty} (-a)^r x^{\alpha r} E_{2\alpha,\alpha+2-\alpha r}^{r+1}(-bx^{2\alpha})\phi(x)$$

Where $E_{\alpha,\beta}^{\gamma}(x)$ is Mittag-Leffler function of three parameter.

If instead of considering $\phi(x)$ Heaviside function, if we consider it as parabolic function, the solution becomes

$$I_c(x) = \frac{C}{L} x^{\alpha+1} \sum_{r=0}^{\infty} (-a)^r x^{\alpha r} E_{2\alpha,\alpha+2+\alpha r}^{r+1}(-bx^{2\alpha})\phi(x)$$

Conclusion:

Concepts of fractional calculus are very hard to understand that's why though the fractional calculus was as old as traditional calculus. It was not developed, discussed and applied for long time. In recent year, last two to three decades it is the topic of most of the researchers. Fractional calculus is an extraordinary and outstanding mathematical topic since it is applied to situations where existing theory fails to apply properly. Fractional calculus is applied to all most all sciences and real life problems as discussed above. So, we may say "Fractional Calculus is the mathematical solution to real life problems".

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